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A KEY

TO

DAY'S ALGEBRA:

CONTAINING

A CORRECT EXPOSITION

OF

ALL THE PROBLEMS AND EXAMPLES.

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PREFACE.

Whatever the public wishes for, the public will have, be the object of that desire rational or irrational, proper or improper. There are men enough, possessed of talent, though not principle, who will pander to such tastes. The author of the following work would very respectfully beg to be exempted from all connection or assimilation with that class. But he will not disguise this treatise under the title of "Key for Instructors," "Aid to Teachers," &c. The originators of such works know that no individual is fit for the situation of teacher, who is obliged to resort to them; and that their circulation is intended, in a great measure, if not wholly, for pupils. But such titles make books popular, make them sell, and therefore they use them, though they laugh in secret at the *gullibility* of the public. We again declare that we will resort to no such deceit. We believe our object to be a good one. If it is good, we should not be afraid to avow it; and if it is bad, the public should be the first to know it. We freely say that this work is intended for scholars—to render some little assistance to that class of persons (not few in number) in our colleges and public institutes, who are distinguished for attainments in belles-lettres and classics, but with no sort of ability or inclination to pursue mathematics.

Individuals of this character, forced by our collegiate systems to pursue this branch, and unprovided with keys to assist them in their difficulties, become disheartened by their inability to understand it, or are wounded in pride by their manifest inferiority in it; and are often induced by this fact to leave college and give up the attainment of an education. Thus many a genius slumbers at the plough or anvil, that should have graced a profession or even our national councils. Were the public supplied with keys to their mathematical works, with an entire series of them too, then this evil in a great measure would be remedied; and this peculiar kind of talent (by no means a mean one) would receive its proper station. We hope this work will be carried on, and the public supplied with works of this character, either by the authors themselves, or other persons of suitable talent. What we have done, we know, is but a drop in the bucket; but we freely tender it for the use of the community, knowing that "every little helps." If we have done wrong in publishing it, all we can say is, that it is an error of the head, and not of the heart.

OMNIBUS JUSTITIAM.

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KEY TO DAY'S ALGEBRA.

ADDITION.

Ex. 1.

To	$+ab+8$
Add	$-3+cd$
"	$+5ab+2 \quad -4m$
Ans.	$+6ab+7+cd-4m.$

Ex. 2.

To	$+x+3y-dx$
Add	$-x \quad +7$
"	$-8+hm$
Ans.	$+3y-dx-1+hm.$

Ex. 3.

To	$+abm-3x+bm$
Add	$-x \quad +y+7$
"	$+5x \quad -6y+9$
Ans.	$+abm+x+bm-5y+16.$

Ex. 4.

To	$+3am+6-7xy$
Add	-8
"	$+5am-9+10xy$
Ans.	$+8am-11+3xy.$

Ex. 5.

To	$+6ahy+7d-1+mxy$
Add	$+3ahy-7d+17-mxy$
Ans.	$+9ahy \quad +16.$

Ex. 6. To $+7ad-h+8xy$
Add $-ad$
" $+5ad+h-7xy$

Ans. $+11ad\quad+xy$.

Ex. 7. To $+3ab-2ay+x$
Add $+ab-ay\quad+bx-h$

Ans. $+4ab-3ay+x+bx-h$.

Ex. 8. To $+2by-3ax+2a$
Add $-by\quad+a+3bx$

Ans. $+by-3ax+3a+3bx$.

SUBTRACTION.

Ex. 2. From $+13ad+xy+d$
Sub. $+7ad-xy+d+hm-ry$

Ans. $+6ad+2xy\quad-hm+ry$.

Ex. 3. From $+7abc-8+7x$
Sub. $+3abc-8\quad-dx+r$

Ans. $+4abc\quad+7x+dx-r$.

Ex. 4. From $+h-2y+3ad$
Sub. $+3h+7y+4ad-mx-hy$
" $-ad$

Ans. $-2h-9y\quad+mx+hy$.

Ex. 5. From $+6am-dy+8$
Sub. $+am+3dy+16$
" $-8-e+r$

Ans. $+5am-4dy\quad+e-r$.

Ex. 6. From $+7ay-2x+5$
Sub. $-ay+x+4+h+3b$
Ans. $+8ay-3x+1-h-3b.$

MULTIPLICATION.

Ex. 1. Mult. $a+3b-2$
By $4a-6b-4$

$$4aa+12ab-8a$$
$$-6ab\quad -18bb+12b$$
$$-4a\quad -12b+8$$

Ans. $4aa+6ab-12a-18bb+8.$

Ex. 2. Mult. $8abx$ *
By $3my-1+h$
Ans. $24abxmy-8abx+8abxh.$

Ex. 3. Mult. $28ah-4y$ †
By $60dx$ ‡
Ans. $1680ahdx-240dxy.$

Ex. 4. Mult. $12ab-2hd+2$ This is the same as $(6ab-hd+1)\times 2$.
By $7d+4dx$ §
Ans. $84abd-14hdd+14d+48abdx-8hddx+8dx.$

Ex. 5. Mult. $3ay+y-4+h$
By $dh+dy+hx+xy$ ‖
Ans. $3aydh+dhy-4dh+hdh+3aydy+ydy-4dy+hdy+3ayhx+yhx-4hx+hhx+3ayxy+yxy-4xy+hxy.$

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* $8abx=4ab\times x\times 2.$ † $28ah-4y=(7ah-y)4.$
‡ $60dx=4x\times 3\times 5\times d.$
§ $7d+4dx=8d+4dx-d=(8+4x-1)\times d.$
‖ $dh+dy+hx+xy=(d+x)\times(h+y).$

Ex. 6. Mult. $6ax-4h+d$
By $b+h+bh+1$ *

Ans. $6axb-4hb+db+6axh-4hh+dh+6axbh-4hbh+dbh+6ax-4h+d$.

Ex. 7. Mult. $7ay-1+hd-hx$ †
By $-r-3+4m$, for this is the same as $-(r+3-4m)$

Ans. $-7ayr+r-hdr+hxr-21ay+3-3hd+3hx+28aym-4m+4hdm-4hxm$.

We will here give an explanation of the principle, "Like signs give *plus*, and unlike signs *minus*." Why does $-$ into $+$ give $-$? Multiplication is but continued addition. Thus $n\times 5$ is the same as $n+n+n+n+n=5n$. Then in multiplying $+n$ by -5; the $+n$, repeated five times, gives us $+5n$. Thus, $+n+n+n+n+n=+5n$. But what does the $-$ before the 5 show? It shows that, after adding the n five times as above, the amount is to be subtracted. But we subtract a quantity by changing its sign (Art. 82): Then the $+5n$ is to be subtracted by changing its sign. Thus $+5n$ becomes $-5n$. 2d. Why does $-n$ into $+5$ give $-5n$? The $-n$ repeated five times gives $-n-n-n-n-n=-5n$. But what effect has the $+$ before the 5? It shows that the $-5n$ is to be added. However, in adding a quantity we do not change its sign, but set it down with its former sign. Thus $-5n$ continues $-5n$. 3d. Why does $-$ into $-$ give $+$? Again, $-n$ repeated five times gives $-n-n-n-n-n=-5n$. But the minus sign before the 5 shows that the $-5n$ is to be subtracted, and we subtract a minus quantity like a plus one by changing its sign. Then the sign of $-5n$ is to be changed. Thus $-5n$ becomes $+5n$. Why $+$ into $+$ gives $+$, needs no explanation.

* $b+h+bh+1=(b+1)\times(h+1)$.

† The $(d-x)$ is only multiplied into h, which gives $hd-hx$.

DIVISION.

The principle of "signs" in Division, is the same as in Multiplication, for Division is proved by Multiplication. The Quotient multiplied by the Divisor should give the Dividend; and the sign of the *quotient* multiplied by the sign of the *divisor*, should give the sign of the *dividend*.

Ex. 1. Div. $12aby+6abx-18bbm+24b$
By $6b$
Ans. $2ay+ax-3bm+4$.

Ex. 2. Div. $16a-12+8y+4-20adx+m$
By 4
Ans. $4a-3+2y+1-5adx+\frac{m}{4}$.

Ex. 3. Div. $(a-2h)\times(3m+y)\times x$
By $(a-2h)\times(3m+y)$
Ans. x.*

Ex. 4. Div. $ahd-4ad+3ay-a$ †
By $hd-4d+3y-1$
Ans. a.

Ex. 5. Div. $ax-ry+ad-4my-6+a$
By $-a$
Ans. $-x+\frac{ry}{a}-d+\frac{4my}{a}+\frac{6}{a}-1$.

Ex. 6. Div. $amy+3my-mxy+am-d$
By $-dmy$
Ans. $-\frac{a}{d}-\frac{3}{d}+\frac{x}{d}-\frac{a}{dy}+\frac{1}{my}$.

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* See Art. 118. † This equals $=a(hd-4d+3y-1)$.

Ex. 7. Div. $ard-6a+2r-hd+6$
By $2ard$

Ans. $\frac{1}{2}-\frac{3}{rd}+\frac{1}{ad}-\frac{h}{2ar}+\frac{3}{ard}$.

Ex. 8. Div. $6ax-8+2xy+4-6hy$
By $4axy$.

Ans. $\frac{3}{2y}-\frac{2}{axy}+\frac{1}{2a}+\frac{1}{axy}-\frac{3h}{2ax}$.

The examples under the head of Fractions, we pass over as so easy and simple, that a key for them is unnecessary and superfluous. A moment's thought will solve any question there.

THE SIMPLE EQUATIONS.

We will here explain a few abbreviations used in this work. Trans. = Transposing. Red. = Reducing. Mult. = Multiplying. Div. = Dividing. L. C. M. = Least Common Multiple. No. = Number.

EXAMPLE 1.

$\frac{3x}{4}+6=\frac{5x}{8}+7$. $8=$L. C. M. Mult. by 8, we have $6x+48=5x+56$. Trans. $6x-5x=56-48$. Red. $x=8$.

EXAMPLE 2.

$\frac{x}{a}+h=\frac{x}{b}-\frac{x}{c}+d$. Clearing of Fractions, $bcx+abch=acx-abx+abcd$. Trans. $bcx+abx-acx=abcd-abch$. But $bcx+abx-acx=x(bc+ab-ac)$. Div. by this coefficient of x, then $x=\frac{abcd-abch}{bc+ab-ac}$.

EXAMPLE 3.

$40-6x-16=120-14x$. Trans. $14x-6x=120+16-40$. Red. $8x=96$. $x=12$.

EXAMPLE 4.

$\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$. $6=$ L. C. M. Mult. by 6, $3x-9+2x=120-3x+57$. Trans. $3x+3x+2x=120+57+9$. Red. $8x=186$. $4x=93$. $x=\frac{93}{4}$.

EXAMPLE 5.

$\frac{x}{3}+\frac{x}{5}=20-\frac{x}{4}$. Mult. by 60, $20x+12x=1200-15x$. Trans. $20x+15x+12x=1200$. Red. $47x=1200$. $x=\frac{1200}{47}$.

EXAMPLE 6.

$\frac{1-a}{x}-4=5$. Trans. $\frac{1-a}{x}=5+4=9$. $1-a=9x$. $\frac{1-a}{9}=x$.

EXAMPLE 7.

$\frac{3}{x+4}-2=8$. Trans. $\frac{3}{x+4}=8+2=10$. Mult. by $x+4$, then $3=10x+40$. Trans. $-40+3=10x$. $-37=10x$. Red. $-\frac{37}{10}=x$.

EXAMPLE 8.

$\frac{6x}{x+4}=1$. $6x=x+4$. Trans. $6x-x=4$. $5x=4$. $x=\frac{4}{5}$.

EXAMPLE 9.

$x+\frac{x}{2}+\frac{x}{3}=11$. Mult. by 6, $6x+3x+2x=66$. Reducing, $11x=66$. $x=6$.

Example 10.

$\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$. $60=$ L. C. M. Mult. by 60, $30x+20x-15x=42$. Red. $35x=42$. Div. by 7, $5x=6$. $x=\frac{6}{5}$.

Example 11.

$\frac{x-5}{4}+6x=\frac{284-x}{5}$. Mult. by 20, $5x-25+120x=1136-4x$ Trans. $120x+5x+4x=1136+25$. Red. $129x=1161$. $x=9$.

Example 12.

$3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$. Mult. by 10, $30x+4x+12=50+55x-185$. Trans. $185+12-50=55x-30x-4x$. Red. $147=21x$. $7=x$.

Example 13.

$\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$. $3=$L. C. M. Mult. by 3, $6x-4-6=18-4x+3x$. Trans. $6x+4x-3x=18+6+4$. Red. $7x=28$. $x=4$.

Example 14.

$21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$. $16=$L. C. M. Mult. by 16, it gives, $336+3x-11=10x-10+776-56x$. Trans. $56x+3x-10x=776+11-336-10$. Red. it makes, $49x=441$. And $x=\frac{441}{49}$, or$=9$.

Example 15.

$3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$. $12=$L. C. M. Mult. by 12, it gives, $36x-3x+12-48=20x+56-1$. Trans. $36x-20x-3x=56+48-12-1$. Red. $13x=91$. And $x=\frac{91}{13}$, or$=7$.

EXAMPLE 16.

$\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$. Clearing of fractions by mult. by 30, $70x+50-96-24x+180=45x+135$. Trans. $70x-45x-24x=135+96-180-50$. Red. $70x-69x=231-230$. And $x=1$.

EXAMPLE 17.

$\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$. 15=L. C. M. Mult. by 15, it gives, $51-9x-20x-10=75-90x+35x+70$. Trans. $90x-35x-20x-9x=75+70+10-51$. Red. we find, $26x=104$. Then $x=4$.

EXAMPLE 18.

$x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$. 70 = L. C. M. Mult. by 70, we have, $70x-42x+42+280=700-35x-60x+80+56x-56$. Trans. we have, $70x+60x+35x-56x-42x=700+80-280-56-42$. Red. we find, $67x=402$. And $x=6$.

EXAMPLE 19.

$\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$. Mult. by 9, $6x+7+\frac{63x-117}{6x+3}=6x+12$. Trans. $\frac{63x-117}{6x+3}=6x-6x+12-7$. Reducing, $\frac{63x-117}{6x+3}=5$. Clearing of fractions, $63x-117=30x+15$. Trans. $63x-30x=15+117$. Red. $33x=132$. Div. by 33, $x=4$.

EXAMPLE 20.

$\frac{5x+4}{2}:\frac{18-x}{4}::7:4$. Changing the Proportion into an equation by (Art. 188), we have $\frac{4(5x+4)}{2}=\frac{7(18-x)}{4}$. Clearing of fractions, $16(5x+4)=14(18-x)$. Div. by 2, $8(5x+4)=7(18-x)$. Expanding, $40x+32=126-7x$. Trans. $40x-7x=126-32$. Reducing, we have $47x=94$. And $x=2$.

Problem 1.

Let $x=$ the price of the watch. Then, by the conditions of the problem, $4x+70-50=220$. Trans. $4x=220+50-70$. $4x=200$. $x=50$.

Problem 2.

Let $x=$ the number. Then $x+\frac{x}{2}-20=\frac{x}{4}$. $4=$ L. C. M. Mult. by 4, we have $4x+2x-80=x$. Trans. $4x+2x-x=80$. Red. $5x=80$. $x=16$.

Problem 3.

Let $x=$ the whole estate. Then $\frac{x}{2}-1000=$ the share of the first son, $\frac{x}{3}-800=$ the share of the second son, and $\frac{x}{4}-600=$ the share of the third son. Then $\frac{x}{2}-1000+\frac{x}{3}-800+\frac{x}{4}-600=x$. $12=$ L. C. M. Mult. by 12, we have, $6x-12000+4x-9600+3x-7200=12x$. Trans. we find, $6x+4x+3x-12x=12000+9600+7200$. Red. $x=28800$.

Problem 4.

Let $48-x=$ the larger part, and $x=$ the smaller part. Then $\frac{x}{4}+\frac{48-x}{6}=9$. $12=$ L. C. M. Mult. by 12, $3x+96-2x=108$. Trans. $3x-2x=108-96$. $x=12=$ the less. $48-x=36=$ the larger.

PROBLEM 5.

Let x=the number. Then $\frac{x+720}{125}=\frac{7392}{462}$. But $\frac{7392}{462}=16$. Then $\frac{x+720}{125}=16$. Mult. by 125, $x+720=2000$. Trans. $x=2000-720=1280$.

PROBLEM 6.

Let x = what he gains or loses. Mark his gains +, and his losses −. Then $x+350-60=200$. Trans. $x=200+60-350$. $x=-90$.

PROBLEM 7.

Let x = the ship's latitude at starting. Mark the degrees of north latitude with +, those of south latitude with −. Then by the conditions of the problem, $x+4-13+17-19=-11$. Red. $x=0$, or the Equator.

PROBLEM 8.

Let x= the number. Then $\frac{x}{12}+x+12=64$. Mult. by 12, we have, $x+12x+144=768$. Trans. $x+12x=768-144$. $13x=624$. And dividing by 13, $x=48$.

PROBLEM 9.

Let x = the whole estate. Then $\frac{x}{4}+200$=the share of the first child, $\frac{x}{5}+340$=the share of the second child, $\frac{x}{6}+300$= the share of the third child, and $\frac{x}{8}+400$=the share of the fourth child. But these several shares make up the whole estate, or x. Then $x=\frac{x}{4}+200+\frac{x}{5}+340+\frac{x}{6}+300+\frac{x}{8}+400$. $x=\frac{x}{4}+\frac{x}{5}+\frac{x}{6}+\frac{x}{8}+1240$. 120 = L. C. M. Mult. by 120, we have, $120x=30x+24x+20x+15x+148800$. Trans. and red. the unknowns, we have $31x=148800$. Div. by 31, $x=4800$.

Problem 10.

Let $x=$ the number. Then, by the conditions of the problem, the equation will be $\frac{x}{5}-40=500-x$. Mult. by 5, we have, $x-200=2500-5x$. Trans. $x+5x=2500+200$. Red. $6x=2700$. Div. by 6, $x=450$.

Problem 11.

Let $x=$ the smaller number. Then $x+40=$ the larger number. And $x : x+40 :: 5 : 6$. Changing to an equation, $6x=5x+200$. Trans. $6x-5x=200$. $x=200$, or the smaller number. $x+40=240$, or the larger number.

Problem 12.

Let $x=$ the amount of C's prize. The amount of B's prize is \$200 and $\frac{1}{3}$ of C's prize; that is, $200+\frac{x}{3}$. But C's prize is equal to both A's and B's prizes, or $x=200+200+\frac{x}{3}$. Mult. by 3, $3x=600+600+x$. Trans. and red. $2x=1200$. $x=600=$amount of C's prize. $200+\frac{x}{3}=400=$the amount of B's prize. Then the amount of the three prizes=\$200,+ \$400,+600,=\$1200.

Problem 13.

Let $x=$the number. Then, $x : 12+3x :: 2 : 9$. Changing to the form of an equation, $9x=24+6x$. Trans. $9x-6x=24$. $3x=24$. And $x=8$.

Problem 14.

Let $x=$ the distance below the fort, where the ship and boat will meet. The ship has to sail over the whole of x; while the boat, being already 13 miles below the fort, has to sail over only $x-13$. Then the ship's rate of sailing, is to the boat's rate of sailing, as the distance to be passed over by the ship, is to the distance passed over by the boat. Or, using the letters and figures, $5:3::x:x-13$. Changing to the form of an equation, $5x-65=3x$. Trans. $5x-3x=65$. $2x=65$. And $x=32\frac{1}{2}$ miles.

Problem 15.

Let $x=$ the number. Then $\frac{x}{6}-\frac{x}{8}=20$. Multiplying by 48, $8x-6x=960$. Reducing, $2x=960$. And $x=480$.

Problem 16.

Let $x=$ one part. Then $2000-x=$ the other part. The proportion then is, $x:2000-x::9:7$. Changing to an equation, $7x=18000-9x$. Trans. and red. $16x=18000$. And $x=1125$, or one. $2000-x=875$, or the other.

Problem 17.

Let $x=$ the sum of money. Then $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94$. Mult. by 60, we have, $20x+15x+12x=5640$. Red. $47x=5640$. Div. by 47, $x=120$.

Problem 18.

Let $x=$ the number of miles A travels. Then B travels the remainder of the 360 miles, or $360-x=$ the number of miles B travels. Then A's hourly progress, is to B's hourly progress, as the whole number of miles A travels, is to the whole number of miles B travels. Or, using letters and figures, $10:8::x:360-x$. Changing to an equation, we shall have $8x=3600-10x$. Trans. $8x+10x=3600$. $18x=3600$. Div. by 18, $x=200$ miles, or A's journey. $360-x=160$ miles, or B's journey.

Problem 19.

Let $x=$ the man's age. Then $\frac{x}{3}+\frac{x}{4}+20=x$, or H's whole age. Mult. by 12, $4x+3x+240=12x$. Trans. $240=12x-4x-3x$. Red. $240=5x$. And div. by 5, $x=48$ years.

PROBLEM 20.

Let $x=$ the number. Then $\frac{x}{4}-\frac{x}{5}=96$. Clearing of fractions by mult. by 20, $5x-4x=1920$. Red. $x=1920$.

PROBLEM 21.

Let $x=$ the length of the post. And $x=\frac{x}{5}+\frac{3x}{7}+13$. Mult. by 35, $35x=7x+15x+455$. Trans. and red. $13x=455$. And $x=35$.

PROBLEM 22.

Let $x=$ the number. Then $\frac{3}{5}(x+10)=66$. Div. by 3, $\frac{1}{5}(x+10)=22$. Multiplying by 5, $x+10=110$. Trans. $x=110-10=100$.

PROBLEM 23.

Let $x=$ the whole number of trees in the orchard. Then $\frac{3x}{4}=$the number of apple-trees, $\frac{x}{10}=$the number of pear-trees, and the number of peach-trees$=20+\frac{x}{8}$. But these three kinds compose all the trees of the orchard. Then $x=\frac{3x}{4}+\frac{x}{10}+20+\frac{x}{8}$. 40=L. C. M. Mult. by 40, $40x=30x+4x+800+5x$. Trans. and red. $x=800$.

PROBLEM 24.

Let $x=$ the number of gallons of wine that he bought. After using himself 7 gallons, of course the remainder was $x-7$; and $\frac{1}{4}$ of the remainder was $\frac{x-7}{4}$. But the price at which he sold it was the same proportionally as that at which he bought it. Consequently $x:94::\frac{x-7}{4}:20$. Changing to an equation, $20x=\frac{94x-658}{4}$. Mult. by 4, $80x=94x-658$.

Trans. $658=94x-80x$. Red. $658=14x$. And div. by 14, we see $47=x$.

Problem 25.

Let $x=$ the common income of A or B. A contracts a debt to the amount of $\frac{1}{7}$ of it, or $\frac{x}{7}$. This debt in ten years would amount to $\frac{10x}{7}$. B, living upon $\frac{4}{5}$ of his income, of course would save every year $\frac{1}{5}$ of it, or $\frac{x}{5}$. This in ten years would amount to $\frac{10x}{5}$. But B in this time saves 160 dollars more than A gets in debt for, consequently $\frac{10x}{5}-\frac{10x}{7}=160$. Div. by 10, $\frac{x}{5}-\frac{x}{7}=16$. Clearing of fractions by mult. by 35, $7x-5x=560$. $2x=560$. And $x=\$280$.

Problem 26.

Let $x=$the father's age. Then $\frac{x}{4}=$the time he was single. $\frac{x}{7}+5=$the time he was married before he had a son. $\frac{x}{2}=$that part of his life during which his son lived. He lived 4 years after his son's death; so his whole life, or $x=\frac{x}{4}+\frac{x}{7}+5+\frac{x}{2}+4$. $28=$ L. C. M. Mult. by 28, we have, $28x=7x+4x+140+14x+112$. Trans. $28x-14x-7x-4x=140+112$. Red. $3x=252$. $x-84$ years.

Problem 27.

Let $x=$the number. Then $\frac{x}{3}+\frac{x}{4}+\frac{2x}{7}=73$. Mult. by 84, we have, $28x+21x+24x=6132$. Red. $73x=6132$. And dividing by 73, $x=84$.

Problem 28.

Let $x=$his income. Then, by the conditions of the problem, $100+\frac{x}{5}=$what he spent. And $35+\frac{x}{2}=$what he had left.

But the two compose his income, so $x=100+\frac{x}{5}+35+\frac{x}{2}$. Mult. by 10, $10x=1000+2x+350+5x$. Trans. we have, $10x-5x-2x=1000+350$. Red. $3x=1350$. $x=450$.

Problem 29.

Let x=the number of pounds of gunpowder. And then we have, $10+\frac{2x}{3}$=number of pounds of nitre. The number of pounds of sulphur$=\frac{x}{6}-4\frac{1}{2}$. The number of pounds of charcoal is equal to $\frac{1}{7}$ of the nitre minus 2 lbs. The nitre$=10+\frac{2x}{3}$, and $\frac{1}{7}$ of the nitre$=\frac{10}{7}+\frac{2x}{21}$. Then the sulphur$=\frac{10}{7}+\frac{2x}{21}-2$. But these three make up the composition, so $x=10+\frac{2x}{3}+\frac{x}{6}-4\frac{1}{2}+\frac{10}{7}+\frac{2x}{21}-2$. Mult. by 42, which is L. C. M. $42x=420+28x+7x-189+60+4x-84$. Trans. $42x-28x-7x-4x=420+60-189-84$. Red. $3x=207$. And $x=69$ pounds.

Problem 30.

Let x=number of gallons of brandy. Then $x+15$=number of gallons of wine. But there is as much water as both wine and brandy, that is, $2x+15$=number of gallons of water. But all three fill the cask. So $146=x+x+15+2x+15$. Trans. we have, $146-15-15=x+x+2x$. Red. $116=4x$. $29=x$, or gallons of brandy. $x+15=44$, or number of gallons of wine. And $2x+15=73$, or number of gallons of water.

Problem 31.

Let x = what A paid. Then B paid three times what A paid, or $3x$. C paid as much as both A and B, or $4x$. And B paid as much as both B and C, or $7x$. Thus $x+3x+4x+7x=4755$. Red. we have, $15x=4755$. And $x=\$317$, or A's. $3x=\$951$, or B's. $4x=\$1268$, or C's. $7x=\$2219$, or D's.

PROBLEM 32.

Let x=first part. Then $x-3$=second part. And $x+10$ =third part. $x-9$=fourth part. And $x+16$=fifth part. But the sum of all the parts is 99. So $x+x-3+x+10+x-9+x+16=99$. Trans. we have, $x+x+x+x+x=99+9+3-16-10$. Red. $5x=85$. $x=17$.

PROBLEM 33.

Let x=number of shillings received by the fourth son. Then,

$x+9=$	"	"	"	third "	And
$x+21=$	"	"	"	second "	
$x+39=$	"	"	"	first "	Then,

also, $x+x+9+x+21+x+39=7x+6$. Trans. we have, $39+21+9-6=7x-x-x-x-x$. Red. $63=3x$. $21=x$. And the whole amount, or $7x+6=153$.

PROBLEM 34.

Let x=the original number in each flock. After selling 39 from one, he had $x-39$ left in that flock; and selling 93 from the other flock, he had $x-93$ left in that. But there were twice as many in one as the other, so $x-39=2(x-93)=2x-186$. We have by trans. $186-39=2x-x$. Reducing, $147=x$.

PROBLEM 35.

Let x=number of days the first express will travel before he is overtaken. But having been gone already 5 days, the second express will only travel $x-5$ days. But the number of days each travels is proportioned to their several rates of travel per day; or, as they both travel the same distance, the number of days of the first multiplied by his rate of travel, equals the number of days of the second multiplied by his rate of travel. Then $x\times 60=(x-5)\times 75$. $60x=75x-375$. Trans. $375=75x-60x$. $375=15x$. $25=x$. But $x-5$=number of days in which the second overtakes him. And $x-5=20$, or required number.

PROBLEM 36.

Let x=C's age. Then $3x$=B's age. And $6x$=A's age. But the sum of their ages is 140, so $x+3x+6x=140$. $10x=140$. $x=14$, or C's age. $3x=42$, or B's age. $6x=84$, or A's age.

PROBLEM 37.

£$6\frac{1}{2}$=130 shillings. £5=100 shillings. We shall use 100, and 130, instead of £5, and £$6\frac{1}{2}$, to get rid of tedious fractions. Let x = the price per yard of either piece of cloth, since it is the same in both. But the whole cost divided by the price per yard gives the number of yards. For instance, if you buy apples to the amount of 20 cents and they are 4 cents apiece, then $\frac{20}{4}=5$, or the number of apples. So here $\frac{130}{x}$ = length of one piece, and $\frac{100}{x}$ = length of the other piece. Then, by the conditions of the problem, we have, $\frac{130}{x}+10 : \frac{100}{x}+10 :: 6 : 5$. Changing to the form of an equation, $5\left(\frac{130}{x}+10\right)=6\left(\frac{100}{x}+10\right)$. Div. by 10, we have, $5\left(\frac{13}{x}+1\right)=6\left(\frac{10}{x}+1\right)$. Expand. $\frac{65}{x}+5=\frac{60}{x}+6$. Trans. $\frac{65}{x}-\frac{60}{x}=6-5$. $\frac{5}{x}=1$. $x=5$. $\frac{130}{x}=\frac{130}{5}=26$, or the length of one piece. $\frac{100}{x}=\frac{100}{5}=20$, or the length of the other piece.

PROBLEM 38.

Let x = the original capital of A or B. At the end of the first year, $x+40$=A's capital, and $x-40$=B's capital. But the second year A loses $\frac{1}{3}$ of his capital, or $\frac{x+40}{3}$; and B gains 2 times what A loses, minus 40, or $\frac{2x+80}{3}-40$.

Then at the end of the second year, we have $x+40-\frac{x+40}{3}=$ A's capital; and $x-40+\frac{2x+80}{3}-40=$B's capital. But B's capital is now equal to twice A's capital. Then we have, $2\left(x+40-\frac{x+40}{3}\right)=x-40+\frac{2x+80}{3}-40=x+\frac{2x+80}{3}-80$. Expanding this side, $2x+80-\frac{2x+80}{3}=x+\frac{2x+80}{3}-80$. Trans. $2x-x+80+80=\frac{2x+80}{3}+\frac{2x+80}{3}=\frac{4x+160}{3}$. Red. $x+160=\frac{4x+160}{3}$. Mult. by 3, $3x+480=4x+160$. Trans. $480-160=4x-3x$. Red. $320=x$.

Problem 39.

Let $x=$ the number. Then, $x+36:x+52::3:4$. Changing to an equation, $4x+144=3x+156$. Transposing, $4x-3x=156-144$. Red. $x=12$, or the required number.

Problem 40.

Let $x=$the price of the harness. Then $2x=$the price of the horse, and $6x=$the price of the chaise. Then $x+2x+6=360$. $9x=360$. And $x=40$, or price of the harness. $2x=80$, or price of the horse. $6x=240$, or price of the chaise.

Problem 41.

Let $x=$number of gallons. $\frac{1}{3}$ of the contents having leaked out, there remained $\frac{2}{3}$, or $\frac{2x}{3}$. Then $\frac{2x}{3}-21=\frac{x}{2}$. Mult. by 6, we have $4x-126=3x$. Trans. and red. $x=126$ gallons.

Problem 42.

Let $x=$ the age of the sixth son; then $x+4=$ the age of the fifth son. $x+8=$ the age of the fourth son; and $x+12=$ the age of the third son. $x+16=$ the age of the second son; and $x+20=$ the age of the first son. Then, by the conditions of the problem, $x+20=3x$. $20=3x-x$. Red. $20=2x$.

And $10=x$, or age of the sixth. $x+4=14$, or age of the fifth. $x+8=18$, or age of the fourth. $x+12=22$, or age of the third. $x+16=26$, or age of the second. $x+20=30$, or age of the first.

Problem 43.

Let $x=$ the greater part. Then $49-x=$ the lesser part. And $x+6:49-x-11::9:2$. Changing to an equation, we shall have, $2x+12=441-9x-99$. Trans. $2x+9x=441-99-12$. Red. $11x=330$. $x=30$, or one. $49-x=19$, or the other part.

Problem 44.

Let $2x=$ one number; then $3x=$ the other number. So $2x+4:3x+4::5:7$. Changing to an equation, we shall have, $14x+28=15x+20$. Trans. $28-20=15x-14x$. Red. $8=x$. And $2x=16$, or one nnmber. $3x=24$, or the other number.

Problem 45.

Let $x=$ number of gallons in one cask; then $3x=$ number of gallons in the other. Then $3x-4=4(x-4)$. Expanding, $3x-4=4x-16$. Trans. we have, $16-4=4x-3x$. And $x=12$, or number of gallons in one cask. $3x=36$, or number of gallons in the other cask.

Problem 46.

Let $68-x=$ the greater part; and $x=$ the lesser part. Then $84-(68-x)=(40-x)\times 3$. Expanding, $84-68+x=120-3x$. Trans. $x+3x=120+68-84$. Red. $4x=104$. $x=26$. $68-x=42$. Then the two parts are, 42 and 26.

Problem 47.

Let $2x=$ the distance from A to B. Then, by the conditions of the problem, $3x=$ the distance from C to D; and the distance from B to C is equal to $\frac{1}{4}$ the distance from A to B $\left(\frac{x}{2}\right)$, added to $\frac{1}{2}$ the distance from C to D $\left(\frac{3x}{2}\right)$: the whole of this to be divided by 3. Then the distance from B to C $=\left(\frac{x}{2}+\frac{3x}{2}\right)\div3=(2x)\div3$, or $\frac{2x}{3}$. Then $2x+3x+\frac{2x}{3}=34$. Mult. by 3, we have, $6x+9x+2x=102$. $17x=102$. And $x=6$. $2x=12$, or distance from A to B. $\frac{2x}{3}=4$, or distance from B to C. $3x=18$, or distance from C to D.

Problem 48.

Let $2x=$ first part, $3x=$ second part, and $4x=$ third part. Then the $\frac{1}{2}$ of the first, the $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, are equal. $2x+3x+4x=36$. $9x=36$. And $x=4$. Then $2x=8$, or the first part. $3x=12$, or the second part. $4x=16$, or the third part.

Problem 49.

We have two formulas for solving this problem; both of which we will give, though we much prefer the latter, on account of its conciseness and brevity. 1st. Let $x=$ his original stock. Then $x-50=$ what he had left after his expenses were paid. Adding $\frac{1}{3}$ of the stock, he had $x-50+\frac{x-50}{3}$ at the end of the first year. Subtracting his yearly expenses, he had $x-50+\frac{x-50}{3}-50$. Adding $\frac{1}{3}$ of his present stock, $x-50+\frac{x-50}{3}-50+\left(x-50+\frac{x-50}{3}-50\right)\div3=$ his stock at the end of the second year. Subtracting his yearly expenses he had then, $x-50+\frac{x-50}{3}-50+\left(x-50+\frac{x-50}{3}-50\right)\div3-50$. Adding $\frac{1}{3}$ of his present stock, he will have at the end of the third year, $x-50+\frac{x-50}{3}-50+\left(x-50+\frac{x-50}{3}-50\right)\div3-50+\left(x-50+\frac{x-50}{3}-50+(x-50+\frac{x-50}{3}-50)\div3-50\right)\div3=2x$, as, by the conditions of the problem, his stock at the end of the third year equals twice his original stock. We have merely expressed the divisors at the end of vinculums, to get rid of tedious fractions. $27 = $ L. C. M. Multiplying by 27, we have, $27x-1350+9x-450-1350+9x-450+3x-150-450-1350+9x-450+3x-150-450+3x-150+x-50-150-450=54x$. Trans. $27x+9x+9x+9x+3x+3x+3x+x-54x=1350+1350+1350+450+450+450+450+450+450+150+150+150+150+50$. Red. $10x=7400$. $x=740$, or his original stock.

2d. Let

$27x+200 =$ his original stock.
$50 =$ his yearly expenses.

$27x+150 =$ what he had left after paying his expenses.
$9x+50 = \frac{1}{3}$ of his present stock, to be added to the principal.

$36x+200 =$ his stock at the end of the first year.
$50 =$ his yearly expenses.

$36x+150 =$ what he had left after paying his expenses.
$12x+50 = \frac{1}{3}$ of his present stock, to be added to the principal.

$48x+200 =$ his stock at the end of the second year.
$50 =$ his yearly expenses.

$48x+150 =$ what he had left after paying his expenses.
$16x+50 = \frac{1}{3}$ of his present stock, to be added to the principal.

$64x+200 =$ his stock at the end of the third year.

But twice his original stock or $54x+400=64x+200$. Trans. $400-200=64x-54x$. $200=10x$. $20=x$. Substituting the value of x, we have, $27x+200=540+200=740$, or his original stock.

We have used the above because it frees the equation of those long and tedious fractions which we have in the other equation. We have employed $27x$, because 27 is the least number that will clear the unknown of fractions; and 200, because it is the *only* number, from which 50 being subtracted, and $\frac{1}{3}$ of the remainder (150) being added to the remainder itself, will give the original number (200). A glance at the two solutions, shows how incontestably superior the latter is to the former.

PROBLEM 50.

Let $x=$ the whole number of men in his army. Then, by the conditions of the problem, $x=\frac{x}{2}+3600+\frac{x}{8}+600+\frac{x}{5}$. $40=$ L. C. M. Mult. by 40, we have, $40x=20+144000+5x+24000+8x$. Transposing, we shall have, $40x-20x-8x-5x=144000+24000$. Red. $7x=168000$. And $x=24000$.

INVOLUTION.

EXAMPLES.

$$\frac{a^3+b}{b^4}\times\frac{a-b}{3}=\frac{(a^3+b)\times(a-b)}{3b^4}=\frac{a^4+ab-a^3b-b^2}{3b^4}. \quad \text{(Ex. 8.)}$$

$$\frac{a^5+1}{x^2}\times\frac{b^3-1}{x+a}=\frac{(a^5+1)\times(b^3-1)}{x^2(x+a)}=\frac{a^5b^3-a^5+b^3-1}{x^3+ax^2}. \quad \text{(Ex. 9.)}$$

$$\frac{b^4}{a^{-2}}\times\frac{h^{-3}}{x}\times\frac{a^n}{y^{-3}}=\frac{b^4h^{-3}a^n}{a^{-2}xy^{-3}}=\frac{b^4h^{-3}a^{n+2}}{xy^{-3}}, \text{ or } \frac{b^4y^3a^{n+2}}{xh^3}. \quad \text{(Ex. 10.)}$$

$$\frac{a^4}{y^3}\div\frac{a^3}{y^2}=\frac{a^4}{y^3}\times\frac{y^2}{a^3}=\frac{a^4y^2}{a^3y^3}, \text{ or } \frac{a}{y}. \quad \text{(Ex. 11.)}$$

$$\frac{a^3-x^4}{a^2}\div\frac{x^2-a^{-2}}{a}=\frac{a^3-x^4}{a^2}\times\frac{a}{x^2-a^{-2}}=\frac{a^4-ax^4}{a^2x^2-a^0}, \text{ or } \frac{a^4-ax^4}{a^2x^2-1}. \quad (12.)$$

$$\frac{b-y^{-1}}{y}\div\frac{a^3+b^{-4}}{y^3}=\frac{b-y^{-1}}{y}\times\frac{y^3}{a^3+b^{-4}}=\frac{by^3-y^2}{a^3y+b^{-4}y}=\frac{by^2-y}{a^3+b^{-4}}. \quad (13.)$$

$$\frac{h^3-1}{d^4}\div\frac{d^n+1}{h}=\frac{h^3-1}{d^4}\times\frac{h}{d^n+1}=\frac{h^4-h}{d^{n+4}+d^4}. \quad \text{(Ex. 14.)}$$

Many simple examples in Involution and Evolution we shall pass over, as we have done with those of a similar character in Addition, Subtraction, Multiplication, &c.; as they do but cumber the work, without being of any use or benefit to the student.

EVOLUTION.

REDUCTION OF RADICAL QUANTITIES.

Ex. 1. $\sqrt{8}=\sqrt{4}\times\sqrt{2}$. But $\sqrt{4}=2$. Then $\sqrt{8}=2\sqrt{2}$.

Ex. 2. $\sqrt{a^2x}=\sqrt{a^2}\times\sqrt{x}$. But $\sqrt{a^2}=a$. Then $\sqrt{a^2x}=a\sqrt{x}$.

Ex. 3. $\sqrt{18}=\sqrt{9}\times\sqrt{2}$. But $\sqrt{9}=3$. Then $\sqrt{18}=3\sqrt{2}$.

Ex. 4. $\sqrt[3]{64b^3c}=\sqrt[3]{64b^3}\times\sqrt[3]{c}$. But $\sqrt[3]{64b^3}=4b$. $\sqrt[3]{64b^3c}=4b\sqrt[3]{c}$.

Ex. 5. $\sqrt[4]{\frac{a^4b}{c^5d}}=\sqrt[4]{\frac{a^4}{c^4}}\times\sqrt[4]{\frac{b}{cd}}$. But $\sqrt[4]{\frac{a^4}{c^4}}=\frac{a}{c}$. $\sqrt[4]{\frac{a^4b}{c^5d}}=\frac{a}{c}\sqrt[4]{\frac{b}{cd}}$.

Ex. 6. $\sqrt[n]{a^nb}=\sqrt[n]{a^n}\times\sqrt[n]{b}$. But $\sqrt[n]{a^n}=a$. Then, $\sqrt[n]{a^nb}=a\sqrt[n]{b}$.

Ex. 7. $(a^3-a^2b)^{\frac{1}{2}}=\sqrt{a^2}\times(a-b)^{\frac{1}{2}}$. But $\sqrt{a^2}=a$. Thus, $(a^3-a^2b)^{\frac{1}{2}}=a(a-b)^{\frac{1}{2}}$.

Ex. 8. $(54a^6b)^{\frac{1}{3}}=\sqrt[3]{27a^6}\times(2b)^{\frac{1}{3}}$. $\sqrt[3]{27a^6}=3a^2$. So $(54a^6b)^{\frac{1}{3}}=3a^2(2b)^{\frac{1}{3}}$.

Ex. 9. $\sqrt{98a^2x}=\sqrt{49a^2}\times\sqrt{2x}$. But $\sqrt{49a^2}=7a$. $\sqrt{98a^2x}=7a\sqrt{2x}$.

Ex. 10. $\sqrt[3]{a^3+a^3b^3}=\sqrt[3]{a^3}\times\sqrt[3]{1+b^3}$. $\sqrt[3]{a^3}=a$. So $\sqrt[3]{a^3+a^3b^3}=a\sqrt[3]{1+b^3}$.

ADDITION AND SUBTRACTION OF RADICAL QUANTITIES.

From $\sqrt{50}$, subtract $\sqrt{8}$. $\sqrt{50}=\sqrt{25}\times\sqrt{2}$. $\sqrt{25}=5$. $\sqrt{50}=5\sqrt{2}$. $\sqrt{8}=\sqrt{4}\times\sqrt{2}$. $\sqrt{4}=2$. $\sqrt{8}=2\sqrt{2}$. Then $\sqrt{50}-\sqrt{8}=5\sqrt{2}-2\sqrt{2}=3\sqrt{2}$.

From $\sqrt[3]{b^4y}$, subtract $\sqrt[3]{by^4}$. $\sqrt[3]{b^4y}=\sqrt[3]{b^3}\times\sqrt[3]{by}$. But $\sqrt[3]{b^3}=b$. Then $\sqrt[3]{b^4y}=b\sqrt[3]{by}$. $\sqrt[3]{by^4}=\sqrt[3]{y^3}\times\sqrt[3]{by}$. But

$\sqrt[3]{y^3}=y$. So $\sqrt[3]{by^4}=y\sqrt[3]{by}$. So $\sqrt[3]{b^4y}-\sqrt[3]{by^4}=b\sqrt[3]{by}-y\sqrt[3]{by}$, which is equal to $(b-y)\times\sqrt[3]{by}$.

From $\sqrt[n]{x}$, subtract $\sqrt[6]{x}$, $=\sqrt[n]{x}-\sqrt[6]{x}$.

MULTIPLICATION OF RADICAL QUANTITIES.

Ex. 1. $\sqrt{a}=\sqrt[6]{a^3}$. $\sqrt[3]{b}=\sqrt[6]{b^2}$. $\sqrt{a}\times\sqrt[3]{b}=\sqrt[6]{a^3}\times\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

Ex. 2. $5\sqrt{5}\times3\sqrt{8}=15\sqrt{40}$. $\sqrt{40}=\sqrt{4}\times\sqrt{10}$. But $\sqrt{4}=2$. Then, $\sqrt{40}=2\sqrt{10}$. And $15\sqrt{40}=15\times2\sqrt{10}$, or $30\sqrt{10}$.

Ex. 3. $2\sqrt{3}=2\sqrt[6]{27}$. $3\sqrt[3]{4}=3\sqrt[6]{16}$. Then $2\sqrt{3}\times3\sqrt{4}=2\sqrt[6]{27}\times3\sqrt[6]{16}=6\sqrt[6]{432}$.

Ex. 4. $\sqrt{d}=\sqrt[6]{d^3}$. $\sqrt[3]{ab}=\sqrt[6]{a^2b^2}$. Then $\sqrt{d}\times\sqrt[3]{ab}=\sqrt[6]{d^3}\times\sqrt[6]{a^2b^2}=\sqrt[6]{a^2b^2d^3}$.

Ex. 5. $\sqrt{\frac{2ab}{3c}}\times\sqrt{\frac{9ad}{2b}}=\sqrt{\frac{18a^2bd}{6bc}}=\sqrt{\frac{3a^2d}{c}}$.

Ex. 6. $a\sqrt{a-x}$, mult. by $(c-d)\times\sqrt{ax}=(ac-xd)\times\sqrt{a^2x-ax^2}$. The rational parts, $a\times(c-d)$, give $(ac-ad)$. The radical parts, $\sqrt{a-x}\times\sqrt{ax}=\sqrt{a^2x-ax^2}$.

DIVISION OF RADICAL QUANTITIES.

Ex. 1. $2\sqrt{bc}=2\sqrt{b^3c^3}$. $3\sqrt{ac}=3\sqrt{a^3c^3}$. Then $2\sqrt{bc}\div3\sqrt{ax}$ is the same as $2\sqrt{b^3c^3}\div3\sqrt{a^3c^3}=\frac{2}{3}\sqrt[6]{\frac{b^3c^3}{a^3c^3}}=\frac{2}{3}\sqrt[6]{\frac{b^3}{a^3c}}$.

Ex. 2. $10\sqrt[3]{108}\div5\sqrt[3]{4}=2\sqrt[3]{27}$. But $\sqrt[3]{27}=3$. So $2\sqrt[3]{27}=2\times3=6$.

Ex. 3. $10\sqrt{27}\div2\sqrt{3}=5\sqrt{9}$. But $\sqrt{9}=3$. Then $5\sqrt{9}=5\times3=15$.

Ex. 4. $8\sqrt{108} \div 2\sqrt{6} = 4\sqrt{18}$. But $\sqrt{18} = \sqrt{9} \times \sqrt{2}$. And $\sqrt{9} = 3$. Then $\sqrt{18} = 3\sqrt{2}$. And $4\sqrt{18} = 4 \times 3\sqrt{2} = 12\sqrt{2}$.

Ex. 5. $\sqrt{d} = \sqrt[6]{d^3}$. Then $(a^2b^2d^3)^{\frac{1}{6}} \div \sqrt[6]{d^3} = (a^2b^2)^{\frac{1}{6}}$, or $(ab)^{\frac{1}{3}}$.

Ex. 6. $2a = \sqrt{4a^2}$. Then $(16a^3 - 12a^2x)^{\frac{1}{2}} \div \sqrt{4a^2} = (4a - 3x)^{\frac{1}{2}}$.

MISCELLANEOUS EXAMPLES.

Ex. 1. $\sqrt[4]{81a^2} = \sqrt{9a}$. But $\sqrt{9a} = \sqrt{9} \times \sqrt{a}$. And $\sqrt{9} = 3$. $\sqrt{9a} = 3\sqrt{a}$.

Ex. 2. The 6th root of $(a+b)^{-3} = (a+b)^{-3 \times \frac{1}{6}} = (a+b)^{-\frac{1}{2}}$.

Ex. 3. The nth root of $(x-y)^{\frac{1}{6}} = (x-y)^{\frac{1}{6} \times \frac{1}{n}} = (x-y)^{\frac{1}{6n}}$.

Ex. 4. $(-125a^3x^6)^{\frac{1}{3}} = -5ax^2$.

Ex. 5. The square root of $\frac{4a^4}{9x^2y^2} = \frac{2a^2}{3xy}$.

Ex. 6. The 5th root of $\frac{32a^5x^{10}}{243} = \frac{2ax^2}{3}$.

Ex. 7. By (Art. 265) the square root of $x^2 - 6bx + 9b^2 = x - 3b$.

Ex. 8. The square root of $a^2 + ay + \frac{y^2}{4} = a + \frac{y}{2}$.

Ex. 9. ax^2, in the form of the 6th root, is $(a^6x^{12})^{\frac{1}{6}}$.

Ex. 10. $-3y$, in the form of the cube root, is $(-27y^3)^{\frac{1}{3}}$.

Ex. 11. a^2 and $a^{\frac{1}{3}}$, under a common index, are $(a^6)^{\frac{1}{3}}$ and $a^{\frac{1}{3}}$.

Ex. 12. $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$, under a common index, are $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$.

Ex. 13. $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$, under the index $\frac{1}{8}$, are $(a^4)^{\frac{1}{8}}$ and $(b^2)^{\frac{1}{8}}$.

Ex. 14. $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$, under the index $\frac{1}{8}$, are $(2^4)^{\frac{1}{8}}$ and $(4^2)^{\frac{1}{8}}$.

Ex. 15. $\sqrt{294}=\sqrt{49}\times\sqrt{6}$. But $\sqrt{49}=7$. Then $\sqrt{294}=7\sqrt{6}$.

Ex. 16. $\sqrt{x^3-a^2x^2}=\sqrt{x^2}\times\sqrt{x-a^2}$. But $\sqrt{x^2}=x$. So $\sqrt{x^3-a^2x^2}=x\sqrt{x-a^2}$.

Ex. 17. $\sqrt{16a^2x}=\sqrt{16a^2}\times\sqrt{x}$. But $\sqrt{16a^2}=4a$. Then $\sqrt{16a^2x}=4a\sqrt{x}$. $\sqrt{4a^2x}=\sqrt{4a^2}\times\sqrt{x}$. But $\sqrt{4a^2}=2a$. Then $\sqrt{4a^2x}=2a\sqrt{x}$. Then the difference of the quantities $=4a\sqrt{x}-2a\sqrt{x}=2a\sqrt{x}$. And the sum of the quantities $=4a\sqrt{x}+2a\sqrt{x}=6a\sqrt{x}$.

Ex. 18. $\sqrt[3]{192}=\sqrt[3]{64}\times\sqrt[3]{3}$. But $\sqrt[3]{64}=4$. Then $\sqrt[3]{192}=4\sqrt[3]{3}$. $\sqrt[3]{24}=\sqrt[3]{8}\times\sqrt[3]{3}$. But $\sqrt[3]{8}=2$. Then $\sqrt[3]{24}=2\sqrt[3]{3}$. Then the difference of the quantities $=4\sqrt[3]{3}-2\sqrt[3]{3}=2\sqrt[3]{3}$. And the sum of the quantities $=4\sqrt[3]{3}+2\sqrt[3]{3}=6\sqrt[3]{3}$.

Ex. 19. $7\sqrt[3]{18}\times5\sqrt[3]{4}=35\sqrt[3]{72}$. But $\sqrt[3]{72}=\sqrt[3]{8}\times\sqrt[3]{9}$. And $\sqrt[3]{8}=2$. Then $\sqrt[3]{72}=2\sqrt[3]{9}$. And $35\sqrt[3]{72}=35\times2\sqrt[3]{9}=70\sqrt[3]{9}$.

Ex. 20. Mult. $4+2\sqrt{2}$
By $2-\sqrt{2}$

$8+4\sqrt{2}$
$-4\sqrt{2}-4$ *

Ans. $8 \qquad -4=4$.

Ex. 21. Mult. $a(a+\sqrt{c})^{\frac{1}{2}}$
By $b(a-\sqrt{c})^{\frac{1}{2}}$

Ans. $ab(a^2-c)^{\frac{1}{2}}$.

The product of the quantities in the vinculums is, by Art. 235, $(a^2-c)^{\frac{1}{2}}$.

Ex. 22. $2(a+b)^{\frac{1}{n}}\times3(a+b)^{\frac{1}{m}}=2(a+b)^{\frac{m}{mn}}\times3(a+b)^{\frac{n}{mn}}=6(a+b)^{\frac{m+n}{mn}}$.

* $\sqrt{2}\times\sqrt{2}=2$. And $2\sqrt{2}\times\sqrt{2}=4$.

Ex. 23. $6\sqrt{54} \div 3\sqrt{2} = 2\sqrt{27}$. But $\sqrt{27} = \sqrt{9} \times \sqrt{3}$. And $\sqrt{9} = 3$. Then $\sqrt{27} = 3\sqrt{3}$. And $2\sqrt{27} = 2 \times 3\sqrt{3} = 6\sqrt{3}$, or answer.

Ex. 24. $4\sqrt[3]{72} \div 2\sqrt[3]{18} = 2\sqrt[3]{4}$.

Ex. 25. $\sqrt{7} = \sqrt[6]{7^3}$. And $\sqrt[3]{7} = \sqrt[6]{7^2}$. Then $\sqrt[6]{7^3} \div \sqrt[6]{7^2} = \sqrt[6]{7}$.

Ex. 26. $8\sqrt[3]{512} \div 4\sqrt[3]{2} = 2\sqrt[3]{256}$. $\sqrt[3]{256} = \sqrt[3]{64} \times \sqrt[3]{4}$. But $\sqrt[3]{64} = 4$. Then $\sqrt[3]{256} = 4\sqrt[3]{4}$. And $2\sqrt[3]{256} = 2 \times 4\sqrt[3]{4} = 8\sqrt[3]{4}$.

Ex. 27. $(17\sqrt{21})^3 = 17\sqrt{21} \times 17\sqrt{21} \times 17\sqrt{21} = 103173\sqrt{21}$.

Ex. 28. The square of $5 + \sqrt{2}$, by Art. 214, is $25 + 10\sqrt{2} + 2$.

Ex. 29. Fourth power of $\frac{1}{6}\sqrt{6}$ is the fourth power of $\frac{1}{6} \times$ fourth power of $\sqrt{6}$. $(\frac{1}{6})^4 = \frac{1}{1296}$. $(\sqrt{6})^4 = 36$. $\frac{1}{1296} \times 36 = \frac{1}{36}$

Ex. 30. The cube of $\sqrt{x} - \sqrt{b} = (\sqrt{x} - \sqrt{b}) \times (\sqrt{x} - \sqrt{b}) \times (\sqrt{x} - \sqrt{b})$. $(\sqrt{x} - \sqrt{b}) \times (\sqrt{x} - \sqrt{b}) = x - 2\sqrt{bx} + b)$. Then mult. by $\sqrt{x} - \sqrt{b}$, again $(x - 2\sqrt{bx} + b) \times (\sqrt{x} - \sqrt{b}) = x\sqrt{x} - 3x\sqrt{b} + 3b\sqrt{x} - b\sqrt{b}$.

Ex. 31. The factor which will make $y^{\frac{1}{3}}$ a rational quantity is $y^{\frac{2}{3}}$. For $y^{\frac{1}{3}} \times y^{\frac{2}{3}} = y^{\frac{1}{3}+\frac{2}{3}} = y^{\frac{3}{3}}$, or y.

Ex. 32. That factor is $\sqrt{5} + \sqrt{x}$. For, by Art. 235, we find that $(\sqrt{5} - \sqrt{x}) \times (\sqrt{5} + \sqrt{x}) = 5 - x$, which is a rational quantity.

Ex. 33. $\frac{\sqrt{a}}{\sqrt{x}} = \frac{\sqrt{a} \times \sqrt{a}}{\sqrt{x} \times \sqrt{a}} = \frac{a}{\sqrt{ax}}$. Here we multiply both numerator and denominator by $\sqrt{a}$, and this gives a rational numerator.

Ex. 34. $\frac{\sqrt{6}}{\sqrt{7} \times \sqrt{3}} = \frac{\sqrt{6}}{\sqrt{21}} = \frac{\sqrt{6} \times \sqrt{21}}{\sqrt{21} \times \sqrt{21}} = \frac{\sqrt{126}}{21}$. Here we multiply by the $\sqrt{21}$, which is the same as $\sqrt{7} \times \sqrt{3}$. This gives a rational denominator.

REDUCTION OF EQUATIONS BY INVOLUTION AND EVOLUTION.

EXAMPLE 1.

$\sqrt{x}+4=9$. Trans. $\sqrt{x}=9-4=5$. Involving, $x=5^2=25$.

EXAMPLE 2.

$a+\sqrt[n]{x}-b=d$. Transposing, $\sqrt[n]{x}=d+b-a$. Involving, $x=(d+b-a)^n$.

EXAMPLE 3.

$\sqrt[3]{x+1}=4$. Involving, $x+1=4^3=64$. Trans. $x=64-1=63$.

EXAMPLE 4.

$4+3\sqrt{x-4}=6\frac{1}{2}$. Trans. $3\sqrt{x-4}=6\frac{1}{2}-4=2\frac{1}{2}$, or $\frac{5}{2}$. Div. by 3, we have $\sqrt{x-4}=\frac{5}{6}$. Involving, $x-4=(\frac{5}{6})^2=\frac{25}{36}$. Trans. $x=\frac{25}{36}+4$.

EXAMPLE 5.

$\sqrt{a^2+\sqrt{x}}=\dfrac{3+d}{\sqrt{a^2+\sqrt{x}}}$. Clearing of fractions, $a^2+\sqrt{x}=3+d$. Trans. $\sqrt{x}=3+d-a^2$. Involving, $x=(3+d-a^2)^2$.

EXAMPLE 6.

$3+2\sqrt{x}-\frac{4}{5}=6$. Trans. $2\sqrt{x}=6+\frac{4}{5}-3=3\frac{4}{5}$, or $\frac{19}{5}$. Div. by 2, $\sqrt{x}=\frac{19}{10}$. Involving, $x=(\frac{19}{10})^2=\frac{361}{100}$.

EXAMPLE 7.

$4\sqrt{\dfrac{x}{5}}=8$. Div. by 4, we have $\sqrt{\dfrac{x}{5}}=2$. Involv. $\dfrac{x}{5}=2^2=4$. Clearing of fractions, $x=20$.

EXAMPLE 8.

$(2x+3)^{\frac{1}{3}}+4=7$. Trans. $(2x+3)^{\frac{1}{3}}=7-4=3$. Involving to the 3d power, $2x+3=3^3=27$. Trans. $2x=27-3=24$. And $x=12$.

Example 9.

$\sqrt{12+x}=2+\sqrt{x}$. Involv. $12+x=(2+\sqrt{x})^2=4+4\sqrt{x}+x$. Trans. $12-4+x-x=4\sqrt{x}$. Red. $8=4\sqrt{x}$. Div. by 4, we have $2=\sqrt{x}$. Involving, $2^2=4=x$.

Example 10.

$\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$. Involving, $x-a=x-\sqrt{ax}+\frac{1}{4}a$, as it is $(\sqrt{x}-\frac{1}{2}\sqrt{a})^2$. Trans. $x-x+\sqrt{ax}=\frac{1}{4}a+a$. Red. $\sqrt{ax}=\frac{5a}{4}$. Involving, we have $ax=\left(\frac{5a}{4}\right)^2=\frac{25a^2}{16}$. Div. by a, then $x=\frac{25a}{16}$.

Example 11.

$\sqrt{5}\times\sqrt{x+2}=2+\sqrt{5x}$. Involving, $5\times(x+2)=(2+\sqrt{5x})^2=4+4\sqrt{5x}+5x$. Expanding, $5x+10=4+4\sqrt{5x}+5x$. Trans. $5x-5x+10-4=4\sqrt{5x}$. Red. $6=4\sqrt{5x}$. Div. by 4, we have $\frac{6}{4}$ or $\frac{3}{2}=\sqrt{5x}$. Involving, $(\frac{3}{2})^2=\frac{9}{4}=5x$. Div. by 5, $\frac{9}{20}=x$.

Example 12.

$\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$. Clearing the first member of the equation by multiplying by $\sqrt{x}$, we have $x-ax=\frac{\sqrt{x}\times\sqrt{x}}{x}=\frac{x}{x}$. But $\frac{x}{x}=1$. Then $x-ax=1$. But $x-ax=(1-a)x$. Then $(1-a)x=1$. And dividing by the coefficient of x, then $x=\frac{1}{1-a}$.

Example 13.

$\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$. Clearing of fractions according to Art. 180, we have, $(\sqrt{x}+28)\times(\sqrt{x}+6)=(\sqrt{x}+38)\times(\sqrt{x}+4)$. Expanding, we find that $x+28\sqrt{x}+6\sqrt{x}+168=x+38\sqrt{x}+4\sqrt{x}+152$. Trans. $x-x+168-152=38\sqrt{x}+4\sqrt{x}-28\sqrt{x}-6\sqrt{x}$. Red. $16=8\sqrt{x}$. And $2=\sqrt{x}$. Involving, $2^2=4=x$.

EXAMPLE 14.

$\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$. Mult. by $\sqrt{a+x}$, then $\sqrt{ax+x^2}+a+x=2a$. Trans. $\sqrt{ax+x^2}=2a-a-x=a-x$. Involving, we shall have $ax+x^2=(a-x)^2=a^2-2ax+x^2$. Trans. $ax+2ax=a^2-x^2+x^2$. Red. $3ax=a^2$. Div. by a, then $3x=a$. And $x=\dfrac{a}{3}$.

EXAMPLE 15.

$x+\sqrt{a^2+x^2}=\dfrac{2a^2}{\sqrt{a^2+x^2}}$. Mult. by $\sqrt{a^2+x^2}$, we shall have, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$. Trans. $x\sqrt{a^2+x^2}=2a^2-a^2-x^2=a^2-x^2$. Involving, $x^2(a^2+x^2)=(a^2-x^2)^2$. Expanding both sides of the equation, $a^2x^2+x^4=a^4-2a^2x^2+x^4$. Trans. $a^2x^2+2a^2x^2=a^4+x^4-x^4$. Red. $3a^2x^2=a^4$. Div. by a^2, we have, $3x^2=a^2$. Extracting the square root, $x\sqrt{3}=a$, and $x=\dfrac{a}{\sqrt{3}}$, or $a\sqrt{\tfrac{1}{3}}$.

EXAMPLE 16.

$a+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$. Involv. $x^2+2ax+a^2=a^2+x\sqrt{b^2+x^2}$. Here $(x+a)^2=x^2+2ax+a^2$. Trans. $x^2+2ax+a^2-a^2=x\sqrt{b^2+x^2}$. Red. $x^2+2ax=x\sqrt{b^2+x^2}$. Div. by x, we have, $x+2a=\sqrt{b^2+x^2}$. Involv. again, $(x+2a)^2=x^2+4ax+4a^2=b^2+x^2$. Trans. and red. $4ax=b^2-4a^2$. Div. by $4a$, then $x=\dfrac{b^2-4a^2}{4a}$.

EXAMPLE 17.

$\sqrt{2+x}+\sqrt{x}=\dfrac{4}{\sqrt{2+x}}$. Mult. by $\sqrt{2+x}$, then $2+x+\sqrt{2x+x^2}=4$. Trans. $\sqrt{2x+x^2}=4-2-x=2-x$. Involving, we shall have, $2x+x^2=(2-x)^2=4-4x+x^2$. Trans. $2x+4x+x^2-x^2=4$. Red. $6x=4$. $x=\tfrac{4}{6}$, or $\tfrac{2}{3}$.

Example 18.

$\sqrt{x-32}=16-\sqrt{x}$. Involving, $x-32=(16-\sqrt{x})^2=256-32\sqrt{x}+x$. Trans. $x-x+32\sqrt{x}=256+32$. Red. $32\sqrt{x}=288$. $\sqrt{x}=9$. Involving again, $x=81$.

Example 19.

$\sqrt{4x+17}=2\sqrt{x}+1$. Involving, $4x+17=(2\sqrt{x}+1)^2=4x+4\sqrt{x}+1$. Trans. $4x-4x+17-1=4\sqrt{x}$. Red. $16=4\sqrt{x}$. $4=\sqrt{x}$. $16=x$.

Example 20.

$\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$. Mult. as in Ex. 13, we shall have, $(\sqrt{6x}-2)\times(4\sqrt{6x}+6)=(4\sqrt{6x}-9)\times(\sqrt{6x}+2)$. $24x+6\sqrt{6x}-8\sqrt{6x}-12=24x+8\sqrt{6x}-9\sqrt{6x}-18$, by expand. Trans. $18-12=24x-24x+8\sqrt{6x}+8\sqrt{6x}-9\sqrt{6x}-6\sqrt{6x}$. Red. $6=\sqrt{6x}$. Involving, $6^2=36=6x$. And $6=x$.

PROBLEMS UNDER THE SAME HEAD.

Problem 1.

Let x = his age. Then $\sqrt{x+10}-2=6$. Trans. we shall have, $\sqrt{x+10}=6+2=8$. Involving, $x+10=8^2=64$. Trans. $x=64-10$. $x=54$, his age.

Problem 2.

Let x = the amount. Then $320 : x :: 5x : 2500$. Changing to an equation, $5x^2=800000$. Div. by 5, we have $x^2=160000$. Extracting the square root, $x=\sqrt{160000}=400$.

Problem 3.

Let x = the number. Then $\sqrt{x+22577}-163=237$. Trans. $\sqrt{x+22577}=237+163=400$. Involving, we shall have, $x+22577=(400)^2=160000$. Trans. $x=160000-22577=137423$.

PROBLEM 4.

Let $x=$ the distance. By the supposition, $x^2-96=48$. Trans. $x^2=48+96=144$. Extracting the square root, $x=\sqrt{144}=12$, or the distance.

PROBLEM 5.

Let $x=$ that number. Then $\frac{3x^2}{4}-12=180$. Trans. $\frac{3x^2}{4}=180+12=192$. Mult. by 4, then $3x^2=768$. Div. by 3, $x^2=256$. Extracting the root, $x=\sqrt{256}=16$.

PROBLEM 6.

Let $x=$ the number. By the conditions of the problem, $8-\frac{x^2}{4}=4$. Trans. $8-4=\frac{x^2}{4}$, and $4=\frac{x^2}{4}$. Multiplying by 4, $16=x^2$. Extracting the root, $\sqrt{16}=4=x$.

PROBLEM 7.

Let $10x=$ the sum of the two numbers. Then $7x=$ the greater number, and $10x-7x=3x=$ the lesser number. By the conditions of the problem, $10x\times 3x=30x^2=270$. Div. by 30, then $x^2=9$. Extracting the root, $x=\sqrt{9}=3$. Thus $3x=9$, or the lesser number. $7x=21$, or the greater number.

PROBLEM 8.

Let $2x=$ the difference of the two numbers. Then, by the conditions of the problem, $9x=$ the larger number. And the larger number minus the difference, that is, $9x-2x=7x$, or the smaller number. Then $(9x)^2-(7x)^2=128$; that is, $81x^2-49x^2=128$. Red. $32x^2=128$. Div. by 32, $x^2=4$. Extracting the root, $x=\sqrt{4}=2$. Thus, $9x=18$, the larger number. $7x=14$, the smaller number.

PROBLEM 9.

Let $x=$ the greater part. Then $18-x=$ the smaller part. And $x^2:(18-x)^2::25:16$. Changing to an equation,

$16x^2=25(18-x)^2$. Extracting the square root, $4x=5(18-x)$. Expanding, $4x=90-5x$. Trans. $5x+4x=9x=90$. Then $x=10$, or one part. $18-x=8$, the other part.

PROBLEM 10.

Let $x=$ the greater part. Then $14-x=$ the lesser part. $\frac{x}{14-x}:\frac{14-x}{x}::16:9$. Changing to an equation, $\frac{9x}{14-x}=\frac{16\times(14-x)}{x}$. Clearing of fractions, $9x^2=16(14-x)^2$. Extracting the square root, $3x=4(14-x)=56-4x$. Trans. and red. $7x=56$. And $x=8$, or one part. $14-x=6$, or the other part.

PROBLEM 11.

Let $5x$ and $4x=$ the two numbers. $(5x)^3=125x^3$. $(4x)^3=64x^3$. Then $125x^3+64x^3=5103$. $189x^3=5103$. $x^3=27$. Extracting the root, $x=\sqrt[3]{27}=3$. $5x=15$, or one number. $4x=12$. or the other number.

PROBLEM 21.

Let $x=$ the number of miles A travelled. Then $x-18=$ the number of miles B travelled. If A could have travelled B's distance $(x-18)$ in $15\frac{3}{4}$ days, then he could have travelled in one day $\frac{x-18}{15\frac{3}{4}}$, which is his daily progress. And if B could have travelled A's distance (x) in 28 days, then he could have travelled in one day $\frac{x}{28}$, which is his daily progress. But the distance each travels is proportioned to his daily progress, so $x:x-18::\frac{x-18}{15\frac{3}{4}}:\frac{x}{28}$. Changing to an equation, $\frac{x^2}{28}=\frac{(x-18)^2}{15\frac{3}{4}}$. Clearing of fractions, $15\frac{3}{4}x^2=28(x-18)^2$. Mult. by 4, then $63x^2=112(x-18)^2$. Div. by 7, we shall have $9x^2=16(x-18)^2$. Extracting the root, $3x=4(x-18)$. Expanding, $3x=4x-72$. Trans. $72=4x-3x=x$, or A's journey. $x-18=54$, or B's journey. $72+54=126$, distance from C to D.

Problem 13.

Let $8x$ and $5x$ = the two numbers. Then $8x\times 5x=360$. $40x^2=360$. And $x^2=9$. Extracting the square root, $x=\sqrt{9}=3$. $8x=24$, or one number. $5x=15$, or the other number.

Problem 14.

Let x = number of yards in one piece; $36-x$ = number of yards in the other piece. Then $(36-x)^2$ = the cost of one piece, and x^2 = the cost of the other. Then $(36-x)^2 : x^2 :: 4 : 1$. Changing to an equation, $(36-x)^2=4x^2$. Extracting the root, $36-x=2x$. $36=3x$. $12=x$, or number of yards in one piece; $36-x=24$, or number of yards in the other.

Problem 15.

Let $3x$ and $2x$ = the two numbers. $(3x)^4=81x^4$, and $(2x)^4=16x^4$. $(3x)^3=27x^3$, and $(2x)^3=8x^3$. Then $81x^4-16^4 : 27x^3+8x^3 :: 26 : 7$. Red. $65x^4 : 35x^3 :: 26 : 7$. Changing to an equation, $455x^4=910x^3$. Div. by 455, then $x^4=2x^3$. Div. by x^3, then $x=2$. $2x=4$, or one number. $3x=6$, or the other number.

Problem 16.

Let x = number of gentlemen; then $x\times x=x^2$ = number of servants, and $x\times 2x^2=2x^3$ = number of dollars. Thus $2x^3=3456$. $x^3=1728$. Extracting the root, $x=\sqrt[3]{1728}=12$, or number of gentlemen.

Problem 17.

Let x = number of companies; then $x\times 4x=4x^2$ = number of men first supplied; and $x\times 3=3x$ = number of men supplied afterwards. Then $4x^2 : 4x^2+3x :: 16 : 17$. Changing to an equation, $68x^2=64x^2+48x$. Trans. $68x^2-64x^2=4x^2=48x$. Div. by $4x$, then $x=12$, or number of companies.

QUADRATIC EQUATIONS, OR EQUATIONS OF THE SECOND DEGREE.

It is well known to the student in Quadratics, that in extracting the root of a binomial or residual square, the middle term disappears; or, in other words, we use only the first and last terms in this operation. This middle term, though important to the value of the equation, is, if we may use the expression, a *dormant* quantity; and needs only to be *expressed*, and not expanded, in order to retain the consistency of the equation. I mean by *expressed*, that, instead of writing out in full the coefficient and letters of the middle term, we use some sign or character to designate the value of this same term. And this is often advantageous, as it saves us the labor of multiplying when we complete the square by the second method, and oftentimes frees us from the trouble of writing out long and tedious expressions. With these introductory remarks, I would here say that, in the following solutions, we shall use the Greek letter π to express the value of the middle term in any particular equation. Thus, in the equation $3x^2-8x=35$. Completing the square by the second method, $36x^2-96x+64=420+64=484$. Extracting the square root, $6x-8=22$. $6x=22+8=30$. And $x=5$. Instead of which, in completing the square, we would write it thus, $36x^2-\pi+64=420+64=484$. Extracting the root, $6x-8=22$. $6x=30$. And $x=5$. The benefit of this practice will be seen in such examples as the 17th, 19th, 27th, and others.

EXAMPLE 1.

$3x^2-9x-4=80$. Trans. $3x^2-9x=80+4=84$. Div. by 3, we have, $x^2-3x=28$. Completing the square, $4x^2-\pi+9=112+9=121$. Extracting the root, $2x-3=\sqrt{121}=11$. Trans. $2x=11+3=14$. Then $x=7$.

EXAMPLE 2.

$4x-\dfrac{36-x}{x}=46$. Mult. by x, then $4x^2-36+x=46x$. Trans.

and red. $4x^2-45x=36$. Completing the square by the second method, $64x^2-*+(45)^2=576=(45)^2=2601$. Extracting the root, $8x-45=\sqrt{2601}=51$. Trans. $8x=51+45=96$ Div. by 8, then $x=12$.

Example 3.

$4x-\frac{14-x}{x+1}=14$. Mult. by $x+1$, then $4x^2+4x-14+x=14x+14$. Trans. $4x^2+4x+x-14x=14+14$. Red. $4x^2-9x=28$. Completing the square by the second method, $64x^2-*+(9)^2=448+(9)^2=529$. Extracting the root, $8x-9=\sqrt{529}=23$. Trans. $8x=23+9=32$. Div. by 8, then $x=4$.

Example 4.

$5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$. Clearing of fractions, we have, $10x^2-30x-6x+6=4x^2-12x+3x^2-9x-6x+18$. Trans. $10x^2-4x^2-3x^2+12x+9x+6x-30x-6x=18-6$. Red. $3x^2-9x=12$. Div. by 3, then $x^2-3x=4$. Completing the square by the second method, $4x^2-*+(3)^2=16+(3)^2=25$. Extracting the root, $2x-3=\sqrt{25}=5$. Trans. $2x=5+3=8$. And $x=4$.

Example 5.

$\frac{16}{x}-\frac{100-9x}{4x^2}=3$. Mult. by $4x^2$, then $64x-100+9x=12x^2$. Trans. $-100=12x^2-64x-9x$. Red. $-100=12x^2-73x$. Completing the square by the second method, we have, $576x^2-*+(73)^2=(73)^2-4800=529$. Extracting the root, $24x-73=\sqrt{529}=23$. Trans. $24x=23+73=96$. Div. by 24, then $x=4$.

Example 6.

$\frac{3x-4}{x-4}+1=10-\frac{x-2}{2}$. Clearing of fractions, $6x-8+2x-8=20x-80-x^2+2x+4x-8$. Transposing, we then have, $x^2+6x+2x-20x-4x-2x=-80+8+8-8$. Red. $x^2-18x=-72$. Completing the square, $x^2-18x+(9)^2=(9)^2-72=9$. Extracting the root, $x-9=\sqrt{9}=3$. And $x=3+9=12$.

EXAMPLE 7.

$\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$. Mult. by 9, then $3x+12-\frac{63-9x}{x-3}$ $=4x+7-9$. Trans. $12+9-7-\frac{63-9x}{x-3}=4x-3x$. Red. $14-\frac{63-9x}{x-3}=x$. Mult. by $(x-3)$, then $14x-42-63+9x=$ x^2-3x. Trans. $-42-63=x^2-3x-14x-9x$. Red. -105 $=x^2-26x$. Completing the square, $(13)^2-105=64=$ $x^2-26x+(13)^2$. Extracting the root, $\sqrt{64}=8=x-13$. And $8+13=21=x$.

EXAMPLE 8.

$\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$. Clearing the first member of the equation of fractions, $x^3-10x^2+1=x^3-6x^2+9x-3x^2+18x-27$. Trans. $x^3-x^3+1+27=10x^2-3x^2-6x^2+9x+18x$. Red. 28 $=x^2+27x$. Completing the square by the second method, $112+(27)^2=841=4x^2+\varpi+(27)^2$. Extracting the root, $\sqrt{841}$ $=29=2x+27$. And $29-27=2=2x$. $1=x$.

EXAMPLE 9.

$\frac{6}{x+1}+\frac{2}{x}=3$. Clearing of fractions, $6x+2x+2=3x^2+3x$. Trans. $2=3x^2+3x-6x-2x$. Red. $2=3x^2-5x$. Completing the square by the second method, $24+(5)^2=49=36x^2-\varpi+$ $(5)^2$. Extracting the root, $\sqrt{49}=7=6x-5$. Trans. $7+5=$ $12=6x$. Div. by 6, then $2=x$.

EXAMPLE 10.

$\frac{3x}{x+2}-\frac{x-1}{6}=x-9$. Mult. by 6, then $\frac{18x}{x+2}-x+1=6x-54$. Trans. $\frac{18x}{x+2}=6x+x-54-1$. Red. $\frac{18x}{x+2}=7x-55$. Mult. by $x+2$, then $18x=7x^2+14x-55x-110$. Trans. $110=$ $7x^2-55x-18x+14x$. Red. $7x^2-59x=110$. Completing the square by the second method, $196x^2-\varpi+(59)^2=3080+(59)^2$ $=6561$. Extracting the root, $14x-59=\sqrt{6561}=81$. Trans. $14x=81+59$. $14x=140$. And $x=10$.

Example 11.

$\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$. Clearing of fractions, $ax^2+a^3=2ax$. Div. by a, then $x^2+a^2=2x$. Trans. $x^2-2x=-a^2$. Completing the square, $x^2-2x+1=1-a^2$. Extracting the root, $x-1=\sqrt{1-a^2}$. Trans. $x=1+\sqrt{1-a^2}$.

Example 12.

$x^4+ax^2=b$. Completing the square, $x^4+ax^2+\frac{a^2}{4}=b+\frac{a^2}{4}$. Extracting the root, $x^2+\frac{a}{2}=\sqrt{b+\frac{a^2}{4}}$. Transposing, $x^2=-\frac{a}{2}+\sqrt{b+\frac{a^2}{4}}$. Extracting the root, $x=\left(-\frac{a}{2}+\sqrt{b+\frac{a^2}{4}}\right)^{\frac{1}{2}}$.

Example 13.

$\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}$. Mult. by 2, then $x^6-\frac{x^3}{2}=-\frac{1}{16}$. Completing the square, $x^6-\frac{x^3}{2}+\frac{1}{16}=\frac{1}{16}-\frac{1}{16}=0$. Extracting the root, $x^3-\frac{1}{4}=0$. Trans. $x^3=\frac{1}{4}$. Extracting the root, $x=\sqrt[3]{\frac{1}{4}}$.

Example 14.

$2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$. Completing the square by the second method, $16x^{\frac{2}{3}}+*+(3)^2=16+(3)^2=25$. Extracting the root, $4x^{\frac{1}{3}}+3=\sqrt{25}=5$. Trans. $4x^{\frac{1}{3}}=5-3=2$. And $x^{\frac{1}{3}}=\frac{1}{2}$. Involving, $x=(\frac{1}{2})^3=\frac{1}{8}$. This example, as well as the two preceding, and the five following ones, are instances of the completion of the square according to Art. 314; by which it will be seen that, in order to complete the square, it only requires that the index of one of the unknowns should be double the other.

Example 15.

$\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$. Mult. by 6, then $3x-2\sqrt{x}=133$. Completing the square by the second method, $36x-*+(2)^2=1596+(2)^2=1600$. Extracting the root, $6\sqrt{x}-2=\sqrt{1600}=40$. Trans. $6\sqrt{x}=40+2=42$. Div. by 6, then $\sqrt{x}=7$. Involving, $x=7^2=49$.

Example 16.

$2x^4-x^2+96=99$. Trans. $2x^4-x^2=99-96=3$. Completing the square by the second method, $16x^4-\pi+1=24+1=25$. Extracting the root, $4x^2-1=5$. $4x^2=5+1=6$. Extracting the root, $2x=\sqrt{6}$. And $x=\frac{1}{2}\sqrt{6}$.

Example 17.

We have two methods for the solution of this example. 1st. Let $y=(10+x)$. Substituting y in the equation, then $y^{\frac{1}{2}}-y^{\frac{1}{4}}=2$. Completing the square by the second method, $4y^{\frac{1}{2}}-\pi+1=8+1=9$. Extracting the root, $2y^{\frac{1}{4}}-1=\sqrt{9}=3$. Trans. $2y^{\frac{1}{4}}=3+1=4$. Div. by 2, then $y^{\frac{1}{4}}=2$. Involving, $y=2^4=16$. But $y=(10+x)$, then $10+x=16$. Trans. $x=16-10=6$. 2d. Dispensing with the y, as one of the vinculated quantities has an index double the other; completing the square by the second method, $4(10+x)^{\frac{1}{2}}-\pi+1=8+1=9$. Extracting the root, $2(10+x)^{\frac{1}{4}}-1=\sqrt{9}=3$. Transposing, $2(10+x)^{\frac{1}{4}}=3+1=4$. Div. by 2, then $(10+x)^{\frac{1}{4}}=2$. Involving, $10+x=2^4=16$. Trans. $x=16-10=6$.

Example 18.

$3x^{2n}-2x^n=8$. Completing the square by the second method, $36x^{2n}-\pi+(2)^2=96+(2)^2=100$. Extracting the root, $6x^n-2=\sqrt{100}=10$. Transposing, $6x^n=10+2=12$. $x^n=2$. And $x=\sqrt[n]{2}$.

Example 19.

We have two methods for solving this example, every way similar to those of Ex. 17. Let $y=(1+x-x^2)$; substituting y, then $2y-\sqrt{y}=-\frac{1}{9}$. Completing the square by the second method, $16y-\pi+1=1-\frac{8}{9}=\frac{1}{9}$. Extracting the root, $4\sqrt{y}-1=\sqrt{\frac{1}{9}}=\frac{1}{3}$. Trans. $4\sqrt{y}=1+\frac{1}{3}=\frac{4}{3}$. Div. by 4, then $\sqrt{y}=\frac{1}{3}$. Involving, $y=\frac{1}{9}$. But $y=1+x-x^2$. Then $\frac{1}{9}=1+x-x^2$. Trans. $x^2-x=1-\frac{1}{9}=\frac{8}{9}$. Completing the square by the second method, $4x^2-x+1=\frac{32}{9}+1=\frac{41}{9}$. Extracting the root, $2x-1=\sqrt{\frac{41}{9}}=\frac{1}{3}\sqrt{41}$. Trans. $2x=\frac{1}{3}\sqrt{41}+1$. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$. Again, as one of the vinculated quantities has an index double the other (Art. 314); completing the square by the second method, $16(1+x-x^2)-x+1=1-\frac{8}{9}=\frac{1}{9}$. Extracting the square root, $4\sqrt{1+x-x^2}-1=\frac{1}{3}$. $4\sqrt{1+x-x^2}=1+\frac{1}{3}=\frac{4}{3}$. Div. by 4, we have, $\sqrt{1+x-x^2}=\frac{1}{3}$. Involving, $1+x-x^2=\frac{1}{9}$. Trans. $x-x^2=\frac{1}{9}-1=-\frac{8}{9}$. Changing signs, $x^2-x=\frac{8}{9}$. Completing the square by the second method, $4x^2-4x+1=1+\frac{32}{9}=\frac{41}{9}$. Extracting the square root, we have, $2x-1=\sqrt{\frac{41}{9}}=\frac{1}{3}\sqrt{41}$. Trans. $2x=1+\frac{1}{3}\sqrt{41}$; and $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$.

EXAMPLE 20.

$\sqrt[3]{x^3-a^3}=x-b$. Involving to the 3d power, $x^3-a^3=(x-b)^3$. Expanding, $x^3-a^3=x^3-3x^2b+3xb^2-b^3$. Trans. we have, $x^3-x^3+3x^2b-3xb^2=a^3-b^3$. Red. $3x^2b-3xb^2=a^3-b^3$. But $3x^2b-3b^2x=3b(x^2-bx)$. Div. by $3b$, then $x^2-bx=\frac{a^3-b^3}{3b}$. Completing the square, $x^2-bx+\frac{b^2}{4}=\frac{a^3-b^3}{3b}+\frac{b^2}{4}$; this last member being brought to a common denominator and reduced, gives $\frac{4a^3-b^3}{12b}$.* Then the equation stands thus: $x^2-bx+\frac{b^2}{4}=\frac{4a^3-b^3}{12b}$. Extracting the root, $x-\frac{b}{2}=\sqrt{\frac{4a^3-b^3}{12b}}$. Trans. we have, $x=\frac{b}{2}+\sqrt{\frac{4a^3-b^3}{12b}}$.

EXAMPLE 21.

$\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. Clearing of fractions, we have, $(\sqrt{4x}+2)$

* To explain the manner in which this quantity is obtained; first, $\frac{a^3-b^3}{3b}+\frac{b^2}{4}=\frac{4a^3-4b^3}{12b}+\frac{3b^3}{12b}$; and reducing this last member, we have $\frac{4a^3-b^3}{12b}$.

$\times\sqrt{x}=(4-\sqrt{x})\times(4+\sqrt{x})$. Expanding, $2x+2\sqrt{x}=16-x$. Trans. and red. $3x+2\sqrt{x}=16$. Completing the square by the second method, $36x+*+4=192+4=196$. Extracting the root, $6\sqrt{x}+2=\sqrt{196}=14$. Trans. $6\sqrt{x}=14-2=12$; and $\sqrt{x}=2$. Involving, $x=4$.

Example 22.

$x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$. Completing the square by the second method, Art. 314 being applicable here, $4x^{\frac{6}{5}}+*+1=3024+1=3025$. Extracting the root, $2x^{\frac{3}{5}}+1=\sqrt{3025}=55$. $2x^{\frac{3}{5}}=55-1=54$. Div. by 2, then $x^{\frac{3}{5}}=27$. Extracting cube root, $x^{\frac{1}{5}}=\sqrt[3]{27}=3$. Involving to the 5th power, $x=3^5=243$.

Example 23.

$\sqrt{2x+1}+2\sqrt{x}=\dfrac{21}{\sqrt{2x+1}}$. A minus sign is placed before the fraction in several editions of this Algebra; this is incorrect. Clearing of fractions, $2x+1+2\sqrt{2x^2+x}=21$. By transposition, $2\sqrt{2x^2+x}=21-1-2x=20-2x$. Div. by 2, then $\sqrt{2x^2+x}=10-x$. Involv. $2x^2+x=(10-x)^2=100-20x+x^2$. Trans. $2x^2-x^2+x+20x=100$. Red. $x^2+21x=100$. Completing the square by the second method, $4x^2+*+(21)^2=400+(21)^2=841$. Extracting the square root, $2x+21=\sqrt{841}=29$. Trans. $2x=29-21=8$. $x=4$.

Example 24.

$2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}$. Clearing of fractions by mult. by $\sqrt{x-a}$, then $2(x-a)+3\sqrt{2x^2-2ax}=7a+5x$. But $2(x-a)=2x-2a$. Transposing this quantity, $3\sqrt{2x^2-2ax}=7a+5x-2x+2a$, or $=9a+3x$. Div. by 3, then $\sqrt{2x^2-2ax}=3a+x$. Inv. we have, $2x^2-2ax=(3a+x)^2=9a^2+6ax+x^2$. Trans. $2x^2-x^2-6ax-2ax=9a^2$. Red. $x^2-8ax=9a^2$. Completing the square, $x^2-8ax+16a^2=9a^2+16a^2$. $9a^2+16a^2=25a^2$. Extracting the root, $x-4a=\sqrt{25a^2}=5a$. Trans. $x=5a+4a$. $x=9a$.

Example 25.

$x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$. Trans. $x+16-10=7\sqrt{x+16}-4\sqrt{x+16}$. Red. $x+6=3\sqrt{x+16}$. Involving both sides, we have, $(x+6)^2=9(x+16)$. Expanding, $x^2+12x+36=9x+144$. Trans. $x^2+12x-9x=144-36$. Red. $x^2+3x=108$. Completing the square by the second method, $4x^2+\pi+(3)^2=432+(3)^2=441$. Extracting the root, $2x+3=\sqrt{441}=21$. Trans. $2x=21-3=18$. Div. by 2, then $x=9$.

Example 26.

$\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$. Div. by $\sqrt{x}$, then $\sqrt{x^4}+\sqrt{x^2}=6$. But $\sqrt{x^4}=x^2$. And $\sqrt{x^2}=x$. Then $x^2+x=6$. Completing the square by the second method, $4x^2+\pi+1=24+1=25$. Extracting the root, $2x+1=\sqrt{25}=5$. Trans. $2x=5-1=4$. Div. by 2, then $x=2$.

Example 27.

$\frac{4x-5}{x}-\frac{3x-7}{3x+7}=\frac{9x+23}{13x}$. Mult. by $13x$, we have now, $52x-65-\frac{39x^2-91x}{3x+7}=9x+23$. Trans. $52x-9x-\frac{39x^2-91x}{3x+7}=23+65$. Red. then $43x-\frac{39x^2-91x}{3x+7}=88$. Mult. by $3x+7$, to clear of fractions, $129x^2+301x-39x^2+91x=264x+616$. Trans. $129x^2-39x^2+301x+91x-264x=616$. Red. $90x^2+128x=616$. Div. by 2, then $45x^2+64x=308$. Completing the square by the second method, $8100x^2+\pi+(64)^2=55440+(64)^2=59536$. Extracting the root, $90x+64=\sqrt{59536}=244$. Trans. $90x=244-64$. Red. $90x=180$. $x=2$.

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Example 28.

$\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$. Mult. by x, which is done by div. the denominators of every term by x, then $\frac{3}{6-x}+\frac{6}{x+2}=\frac{11}{5}$. Clearing of fractions, $15x+30+180-30x=132+44x-11x^2$. Trans. $11x^2+15x-30x-44x=132-180-30$. Red. $11x^2-59x=-78$. Completing the square by the second method, $484x^2-\ast+(59)^2=(59)^2-3432=49$. Extracting the root, $22x-59=\sqrt{49}=7$. Trans. $22x=7+59$. Red. $22x=66$. Div. by 22, then $x=3$.

Example 29.

$(x-5)^3-3(x-5)^{\frac{3}{2}}=40$. Completing the square by the second method, $4(x-5)^3-\ast+(3)^2=160+(3)^2=169$. Extracting the root, $2(x-5)^{\frac{3}{2}}-3=\sqrt{169}=13$. Trans. $2(x-5)^{\frac{3}{2}}=13+3=16$. Div. by 2, then $(x-5)^{\frac{3}{2}}=8$. Evolving, $(x-5)^{\frac{1}{2}}=\sqrt[3]{8}=2$. Involving, $x-5=2^2=4$. Trans. $x=4+5$. And $x=9$.

Example 30.

$x+\sqrt{x+6}=2+3\sqrt{x+6}$. Trans. $x-2=3\sqrt{x+6}-\sqrt{x-6}$. Red. $x-2=2\sqrt{x+6}$. Involving, $(x-2)^2=4(x+6)$. Expanding, $x^2-4x+4=4x+24$. Trans. $x^2-4x-4x=24-4$. Red. $x^2-8x=20$. Completing the square, $x^2-8x+16=20+16=36$. $x-4=\sqrt{36}=6$. Trans. $x=6+4$, or $x=10$.

PROBLEMS UNDER THE SAME HEAD.

Problem 1.

Let $x=$ number of yards of silk. Then $110-x=$ number of yards of cotton. By the conditions of the problem, $400=80(110-x)-x^2$. Expanding, $400=8800-80x-x^2$. Trans. $x^2+80x=8800-400$. Red. $x^2+80x=8400$. Completing the square, $x^2+80x+1600=8400+1600$. But $8400+1600=10000$. Extracting the root, $x+40=\sqrt{10000}=100$. Trans. $x=100-40=60$, or number of yards of silk. $110-x=50$, or number of yards of cotton.

Problem 2.

Let $x=$ age of the eldest brother; then $45-x=$ age of the younger. By the supposition, $x(45-x)=500$. Expanding, $45x-x^2=500$. Changing all the signs of the equation, $x^2-45x=-500$. Completing the square by the second method, $4x^2-x+(45)^2=(45)^2-2000=25$. Extracting the root, $2x-45=\sqrt{25}=5$. Trans. $2x=5+45=50$. Div. by 2, then $x=25$, or age of the eldest. $45-x=20$, or age of the youngest.

Problem 3.

Let $x=$ one number; then $x+4=$ the other number. By the conditions of the problem, $(x+4)x=117$. Expanding, $x^2+4x=117$. Completing the square, $x^2+4x+4=117+4=121$. Extracting the root, $x+2=\sqrt{121}=11$. Trans. $x=11-2=9$, or one number. $x+4=13$, or the other number.

Problem 4.

Let $x=$ his gain; then $30+x$ was the price for which he sold his cloth. By the supposition, $(30+x)x=x^3$. Dividing the equation by x, then $30+x=x^2$. Trans. $-x^2+x=-30$. Changing the signs, $x^2-x=30$. Completing the square by the second method, $4x^2-x+1=120+1=121$. Extracting the root, $2x-1=\sqrt{121}=11$. Trans. $2x=11+1=12$. And $x=6$.

Problem 5.

Let $x=$ the less number; then $x+3=$ the greater number. According to the statement, $(x+3)^3-x^3=117$. Expanding, $x^3+9x^2+27x+27-x^3=117$. Trans. $9x^2+27x+x^3-x^3=117-27$. Red. $9x^2+27x=90$. Div. by 9, then $x^2+3x=10$ Completing the square by the second method, $4x^2+x+(3)^2=$

$40+(3)^2=49$. Extracting the root, $2x+3=\sqrt{49}=7$. Trans. $2x=7-3=4$. And $x=2$, or less number. $x+3=5$, or greater number.

Problem 6.

Let $x=$ one number; then $x+12=$ the other number. By the supposition, $x^2+(x+12)^2=1424$. Expanding, $x^2+x^2+24x+144=1424$. Trans. $x^2+x^2+24x=1424-144$. Red. $2x^2+24x=1280$. Div. by 2, $x^2+12x=640$. Completing the square, $x^2+12x+36=640+36=676$. Extracting the root, $x+6=\sqrt{676}=26$. Trans. $x=26-6=20$, or one number. $x+12=32$, or the other number.

Problem 7.

Let $x=$ the less prize; then $x+120=$ the greater prize. By the statement, $x+120:x::x:10$. Changing to an equation, $10x+1200=x^2$. Trans. $x^2-10x=1200$. Completing the square, $x^2-10x+25=1200+25=1225$. Extracting the root, $x-5=\sqrt{1225}=35$. Trans. $x=35+5=40$, or the less number. $x+120=160$, or greater number.

Problem 8.

Let $x=$ one number; then $6-x=$ the other. By the statement, $x^3+(6-x)^3=72$. Exp. $x^3+216-108x+18x^2-x^3=72$. Trans. $x^3-x^3+18x^2-108x=72-216$. Red. $18x^2-108x=-144$. Div. by 18, then $x^2-6x=-8$. Completing the square, $x^2-6x+9=9-8=1$. Extracting the root, $x-3=1$. $x=1+3=4$, or one number. $6-x=2$, or the other number.

Problem 9.

Let $x=$ one part; then $56-x=$ the other part. By the conditions of the problem, $x(56-x)=640$. Expanding, $56x-x^2=640$. Changing the signs, $x^2-56x=-640$. Completing the square, $x^2-56x+784=784-640=144$. Extracting the root, $x-28=\sqrt{144}=12$. Trans. $x=12+28=40$, or one part. $56-x=16$, or the other part.

Problem 10.

Let $x=$ the number of pieces; then $\frac{675}{x}=$ the cost of one piece, and $48x=$ the amount for which he sold his cloth. By the supposition, $48x=675+\frac{675}{x}$. Div. by 3, then $16x=225+\frac{225}{x}$. Mult. by x, then $16x^2=225x+225$. Trans. $16x^2-225x=225$. Completing the square by the second method, $1024x^2-*+(225)^2=(225)^2+14400=65025$. Extracting the root, $32x-225=255$. Trans. $32x=255+225=480$. Div. by 32, then $x=15$, or number of pieces.

Problem 11.

Let $x=$ B's hourly progress. Then $x+3=$ A's hourly progress. And $\frac{150}{x}=$ number of hours B travels. $\frac{150}{x+3}=$ number of hours A travels. But B travels 8 hours and 20 minutes, or $8\frac{1}{3}$ hours, more than A does. Thus $\frac{150}{x+3}+8\frac{1}{3}=\frac{150}{x}$. Clearing of fractions, $150x+8\frac{1}{3}x^2+25x=150x+450$. Trans. $8\frac{1}{3}x^2+25x+150x-150x=450$. Red. $8\frac{1}{3}x^2+25x=450$. Div. by $8\frac{1}{3}$, or $\frac{25}{3}$, then $x^2+3x=54$. Completing the square by the second method, $4x^2+*+(3)^2=216+(3)^2=225$. Extracting the root, $2x+3=\sqrt{225}=15$. Trans. $2x=15-3=12$. Then $x=6$, or B's hourly progress. $x+3=9$, or A's hourly progress.

Problem 12.

Let $x=$ less number; then $x+6=$ the greater number. By the conditions of the problem, $2x^2+47=(x+6)^2=x^2+12x+36$. Trans. $2x^2-x^2-12x=36-47$. Red. $x^2-12x=-11$. Completing the square, $x^2-12x+36=36-11=25$. Extracting the root, $x-6=\sqrt{25}=5$. Trans. $x=5+6=11$, or the less number. $x+6=17$, or the greater number.

Problem 13.

Let x = number of persons B relieved; then $x+40$ = number of persons A relieved. And $\frac{1200}{x+40}$ = number of dollars A gave to each person. So $\frac{1200}{x}$ = number of dollars B gave to each person. By the supposition, $\frac{1200}{x+40}+5=\frac{1200}{x}$. Clearing of fractions, $1200x+5x^2+200x=1200x+48000$. Trans. $5x^2+200x+1200x-1200x=48000$. Red. $5x^2+200x=48000$. Div. by 5, then $x^2+40x=9600$. Completing the square, $x^2+40x+400=9600+400=10000$. Extracting the root, $x+20=\sqrt{10000}=100$. Trans. $x=100-20=80$, or number B relieved. $x+40=120$, or number A relieved.

Problem 14.

Let x = one number; then $10-x$ = the other number. By the statement, $x^2+(10-x)^2=58$. Expanding, $x^2+100-20x+x^2=58$. Trans. $x^2+x^2-20x=58-100$. Red. $2x^2-20x=-42$. Div. by 2, then $x^2-10x=-21$. Completing the square, $x^2-10x+25=25-21=4$. Extracting the root, $x-5=\sqrt{4}=2$. Trans. $x=2+5=7$, or one number. $10-x=3$, or the other number.

Problem 15.

Let x = number of gentlemen; then $\frac{175}{x}$ = what each one would have paid at first, and $\frac{175}{x-2}$ = what each one afterwards paid. By the supposition, $\frac{175}{x}+10=\frac{175}{x-2}$. Clearing of fractions, $175x-350+10x^2-20x=175x$. Trans. $10x^2-20x+175x-175x=350$. Red. $10x^2-20x=350$. Div. by 10, then $x^2-2x=35$. Completing the square, $x^2-2x+1=35+1=36$. Extracting the root, $x-1=\sqrt{36}=6$. Trans. $x=6+1=7$, or number of gentlemen.

PROBLEM 16.

Let $x=$ number of yards purchased. \$60=6000 cents. \$54=5400 cents. Then $\frac{6000}{x}=$ the price per yard at which he bought the cloth. And $\frac{5400}{x-15}=$ the price per yard at which he sold the cloth. By the statement, $\frac{6000}{x}+10=\frac{5400}{x-15}$. Clearing of fractions, $6000x-90000+10x^2-150x=5400x$. Trans. $10x^2+6000x-5400x-150x=90000$. Red. $10x^2+450x=90000$. Div. by 10, then $x^2+45x=9000$. Completing the square by the second method, $4x^2+x+(45)^2=36000+(45)^2=38025$. Extracting the root, $2x+45=\sqrt{38025}=195$. Trans. $x=195-45$. Red. $2x=150$. And $x=75$, or number of yards purchased. $\frac{6000}{x}=80$, or price per yard of the cloth when purchased.

PROBLEM 17.

Let $x=$ number of days A and B travelled before meeting. Then $x-3=$ B's daily rate of travel. But the daily travel of either, multiplied by the number of days he travelled, gives the distance each has gone. Then $9x=$ A's journey, and $x(x-3)$, or $x^2-3x=$ B's journey. By the supposition, $9x+x^2-3x=247$. Red. $x^2+6x=247$. Completing the square, $x^2+6x+9=247+9=256$. Extracting the root, $x+3=\sqrt{256}=16$. Trans. $x=16-3$, and $x=13$. $9x=117$ miles, or A's travel. $247-117=130$ miles, or B's travel, as he travels the remainder. Or, B's journey $=x(x-3)=10\times x=130$.

PROBLEM 18.

£18=360 shillings, and £16=320 shillings. Let $x=$ price per yard of the coarser piece. So $x+4=$ price per yard of the finer piece. But the whole cost of either piece, divided by the price of it per yard, will give the number of yards in the piece. Thus $\frac{360}{x+4}=$ number of yards in the finer piece. And

$\frac{320}{x}$ = number of yards in the coarser piece. By the conditions of the problem, $\frac{360}{x+4}+2=\frac{320}{x}$. Clearing of fractions, $360x+2x^2+8x=320x+1280$. Trans. $2x^2+360x+8x-320x=1280$. Red. $2x^2+48x=1280$. Div. by 2, then $x^2+24x=640$. Completing the square, $x^2+24x+144=640+144=784$. Extracting the root, $x+12=\sqrt{784}=28$. Trans. $x=28-12=16$, or price of the coarser piece. $x+4=20$, or price of the finer piece per yard. $\frac{320}{x}=20$, or number of yards in the coarser piece. $\frac{360}{x+4}=18$, or number of yards in the finer piece.

Problem 19.

£28+16 shillings = 576 shillings. Let $2x$ = number of gallons of Teneriffe. Then x = cost of the Madeira per gallon, and $x-4$ = cost of the Teneriffe per gallon. $54x$ = whole cost of Madeira, and $2x(x-4)$ = whole cost of Teneriffe. Thus the whole number of gallons in the mixture is $54+2x$. By the supposition, $54x+2x(x-4)=10(54+2x)+576$. Expanding, $54x+2x^2-8x=540+20x+576$. Trans. $2x^2+54x-20x-8x=576+540$. Red. $2x^2+26x=1116$. Div. by 2, then $x^2+13x=558$. Completing the square by the second method, $4x^2+\pi+(13)^2=2232+(13)^2=2401$. Extracting the root, $2x+13=\sqrt{2401}=49$. Trans. $2x=49-13=36$, or number of gallons of Teneriffe, and $x=18$, or cost of Madeira per gallon.

Problem 20.

Let x = the number. By the conditions of the problem, $4=\frac{(\sqrt{40-x^2}+10)2}{x}$. Mult. by x, then $4x=(\sqrt{40-x^2}+10)2$. Dividing by 2, then $2x=\sqrt{40-x^2}+10$. Trans. $2x-10=\sqrt{40-x^2}$. Involving, $(2x-10)^2=40-x^2$. Expanding, $4x^2-40x+100=40-x^2$. Trans. $4x^2+x^2-40x=40-100$.

Red. $5x^2-40x=-60$. Div. by 5, then $x^2-8x=-12$. Completing the square, $x^2-8x+16=16-12=4$. Extracting the root, $x'-4=2$. And $x=2+4=6$, or the number.

Problem 21.

Let $x=$ the number. By the statement, $\frac{x}{2}+\sqrt{x}-12=0$. Trans. $\frac{x}{2}+\sqrt{x}=12$. Mult. by 2, then $x+2\sqrt{x}=24$. Completing the square, $x+2\sqrt{x}+1=24+1=25$. Extracting the root, $\sqrt{x}+1=\sqrt{25}=5$. Trans. $\sqrt{x}=5-1=4$. Involving, $x=4^2=16$.

Problem 22.

Let $x=$ number of gallons in the smaller cask. Then $x+5$ $=$ number of gallons in the larger cask. Thus $2x+5=$ number of gallons in both casks. But $\frac{x}{3}-2=$ the price per gallon. But the price per gallon, multiplied by the number of gallons, gives the whole cost of the wine; thus $(2x+5)\times\left(\frac{x}{3}-2\right)=58$. Expanding, $\frac{2x^2}{3}+\frac{5x}{3}-4x-10=58$. Mult. by 3, then $2x^2+5x-12x-30=174$. Trans. $2x^2+5x-12x=174+30$. Red. $2x^2-7x=204$. Completing the square by the second method, $16x^2-\pi+(7)^2=1632+(7)^2=1681$. Extracting the root, $4x-7=\sqrt{1681}=41$. Trans. $4x=41+7=48$. Div. by 4, then $x=12$, or number of gallons in the smaller cask. $x+5=$ number of gallons in the larger cask. The price per gallon $=\frac{x}{3}-2=2$.

Problem 23.

Let $x=$ number of copper coins. Then $24-x=$ number of silver coins. $x(24-x)=$ whole worth of the copper coins; and $(24-x)x=$whole worth of the silver coins. By the statement, $x(24-x)+(24-x)x=216$. Expanding, $24x-x^2+24x-x^2=216$. Red. $48x-2x^2=216$. Div. by 2, then $24x-x^2=108$.

Changing the signs of all the terms, $x^2-24x=-108$. Completing the square, $x^2-24x+144=144-108=36$. Extracting the root, $x-12=\sqrt{36}=6$. $x=6+12=18$, or number of copper coins. $24-x=6$, or number of silver coins.

Problem 24.

Let $x=$ number of oxen. Thus $\frac{80}{x}=$ cost of each ox. Had he bought 4 more for the same money, then $\frac{80}{x+4}=$ cost of each ox. By the supposition, $\frac{80}{x+4}+1=\frac{80}{x}$. Clearing of fractions, $80x+x^2+4x=80x+320$. Trans. $x^2+4x+80x-80x=320$. Red. $x^2+4x=320$. Completing the square, $x^2+4x+4=320+4=324$. Extracting the root, $x+2=\sqrt{324}=18$. $x=18-2=16$, or number of oxen.

TWO OR MORE UNKNOWN QUANTITIES.

Problem 1.

Let $x=$ the greater number. $y=$ the less. Then, by the conditions,

1st. $x+y=24$.
2d. $x=5y$.

Trans. y in the 1st, $x=24-y=5y$. 2d. Trans. $24=6y$. Red. $4=y$, and $x=5y=20$.

Problem 2.

Let $x=$ the greater number; and $y=$ the less. Then we have,

1st. $x+y=h$.
2d. $x^2-y^2=d$.

Trans. y in the 1st, $x=h-y$. Trans. y^2 in the 2d, $x^2=d+y^2$. Extracting the root, $x=\sqrt{d+y^2}$. Then $\sqrt{d+y^2}=h-y$. Involving, $d+y^2=h^2-2hy+y^2$. Trans. and red. $2hy=h^2-d$, and $y=\frac{h^2-d}{2h}$.

PROBLEM 3.

1st. $ax+by=h.$
2d. $x+y=d.$
Trans. in the 1st, $ax=h-by$, and $x=\frac{h-by}{a}$.

2d. Trans. y, then $x=d-y=\frac{h-by}{a}$. Mult. by a, then $ad-ay=h-by$. Trans. $by-ay=h-ad$, and $y=\frac{h-ad}{b-a}$.

PROBLEM 4.

Let x = the distance the privateer sails; and y = the distance which the ship sails. Then, by the supposition,

1st. $x=y+20.$
2d. $x:y::8:7.$
2d. $7x=8y$. Substituting here the value of x in the 1st, then $7(y+20)=8y$. Expanding, $7y+140=8y$. Then reducing, $y=140$; and $x=y+20=160$.

PROBLEM 5.

Let x = A's age; and y = B's age. By the conditions of the problem,

1st. $x-7=3(y-7)=3y-21.$
2d. $x+7=2(y+7)=2y+14.$
Transposing 7 in the 1st, $x=3y-21+7=3y-14$. Trans. 7 in the 2d, $x=2y+14-7=2y+7$. Thus $3y-14=2y+7$. Trans. $3y-2y=14+7$. And $y=21$.

PROBLEM 6.

Let x = the greater number, and y = the less. Then

1st. $x:y::3:2.$
2d. $x+y=\frac{xy}{6}.$
1st. $2x=3y$. $y=\frac{2x}{3}$. Substituting this value of x in the 2d equation, $x+\frac{2x}{3}=\frac{x}{6}\times\frac{2x}{3}=\frac{2x^2}{18}$. Div. by x, then $1+\frac{2}{3}=\frac{2x}{18}$. Clearing of fractions, $54+36=6x$. Red. $6x=90$. $x=15$. But $y=\frac{2x}{3}=\frac{30}{3}=10$. So 10 and 15 are the numbers.

PROBLEM 7.

Let x = the greater number, and y = the less. Then we have,

1st. $x+y=21110.$ } Mult. 1st by 3, then $3x+3y=63330.$
2d. $2x+3y=52219.$ } Subtracting the 2d equation from this, then $x=11111.$

PROBLEM 8.

1st. $2x+y=16.$ } Mult. the 1st by 3, then $6x+3y=48.$
2d. $3x-3y=6.$ } Adding to this the 2d, $3x-3y=6.$

We then have, $9x=54,$

And $x=6.$

PROBLEM 9.

$x+y=14.$ $x-y=2.$ Subtracting the last equation from the first, then $2y=12$; and $y=6.$

PROBLEM 10.

Let $x=$ the lower part; and $y=$ the upper part.

1st. $\frac{x}{3}+\frac{y}{6}=28.$ } Multiplying the 1st by 15, then $5x+\frac{5y}{2}$
2d. $5x-6y=12.$ } $=420.$

PROBLEM 11.

Let $x=$ the numerator; and $y=$ the denominator. By the conditions,

1st. $\frac{x+1}{y}=\frac{1}{3}.$ } Clearing of fractions in the 1st, $3x+3=y.$
2d. $\frac{x}{y+1}=\frac{1}{4}.$ } Also in the 2d, $4x=y+1.$ Trans. $4x-1=y.$ Making the two values of y equal, then $3x+3=4x-1.$ Trans. $3+1=4x-3x.$ Red. $4=x=$ numerator. Substituting this value of x in the equation, $3x+3=y.$ Then $12+3=y,$ or $15=y=$ denominator.

PROBLEM 12.

Let $x=$ one number; and $y=$ the other. By its conditions, the problem stands,

1st. $x-y:x+y::2:3.$ } Changing the 1st to the form of
2d. $x+y:xy::3:5.$ } an equation, $3x-3y=2x+2y.$

Trans. $3x-2x=2y+3y$. Red. $x=5y$. Changing the 2d to an equation, $5x+5y=3xy$. Substituting the value of x in this equation, $25y+5y=3(5y)y=15y^2$. $30y=15y^2$. Div. by $15y$, then $2=y$. $x=5y=10$. 10 and 2 = the numbers.

Problem 13.

Let x = greater number; and y = less number. By the supposition,

1st. $(x+y)\times(x-y)=5$.
2d. $(x^2+y^2)\times(x^2-y^2)=65$. } Red. 1st, $x^2-y^2=5$. Red. 2d, $x^4-y^4=65$. Div. the 2d by the 1st, then $x^2+y^2=13$, and $x^2=13-y^2$. By trans. in the 1st, $x^2=5+y^2$. Then $5+y^2=13-y^2$. Trans. $2y^2=13-5=8$. $y^2=4$. $y=2=$ the less. Substituting in the 1st, $x^2-4=5$. $x^2=4+5=9$. $x=\sqrt{9}=3=$ the greater number.

Problem 14.

Let x and y = the two numbers. By the conditions,

1st. $x-y=8$.
2d. $xy=240$. } Trans. y in the 1st, $x=8+y$. Div. by y in the 2d, $x=\frac{240}{y}$. Then $8+y=\frac{240}{y}$. Clearing of fractions, $y^2+8y=240$. Completing the square, $y^2+8y+16=256$. Extracting the square root, $y+4=\sqrt{256}=16$. Trans. $y=16-4=12$. Substituting this value in the 1st, $x=8+12=20$. 12 and 20 are the two numbers.

Problem 15.

Let x and y = the two numbers. By the supposition,

1st. $x-y=12$.
5d. $x^2+y^2=1424$. } Trans. y in the 1st, $x=y+12$. Squaring, $x^2=y^2+24y+144$. Trans. y^2 in the 2d, then $x^2=1424-y^2$. Making the two equations equal, $y^2+24y+144=1424-y^2$. Trans. and reducing, $2y^2+24y=1424-144=1280$. Div. by 2, then $y^2+12y=640$. Completing the square, $y^2+12y+36=676$. Extracting the square root, $y+6=\sqrt{676}=26$. $y=26-6=20$. But by the 1st, $x=y+12$; then $x=20+12=32$. 20 and 32 = the two numbers.

Problem 16.

1st. $x+5y+6z=53.$
2d. $x+3y+3z=30.$
3d. $x+y+z=12.$

Subtracting the 2d from the 1st, and $2y+3z=23$. Subtracting the 3d from the 2d, $2y+2z=18$. Subtracting the latter $[2y+2z=18]$ from the former $[2y+3z=23]$, and $z=5$. But $2y+2z=18$. Then $2y+10=18$. $2y=18-10=8$. $y=4$. Again, in the 3d, $x+y+z=12$. Substituting, $x+4+5=12$. $x=12-4-5=3$.

Problem 17.

1st. $x+y+z=12.$
2d. $x+2y+3z=20.$
3d. $\frac{x}{3}+\frac{y}{2}+z=6.$

Subtracting the 1st from the 2d, $y+2z=8$. Mult. the 3d by 3, then $x+\frac{3y}{2}+3z=18$. Subtracting the first from this equation, then $\frac{y}{2}+2z=6$. Subtracting this equation from $[y+2z=8]$, then $\frac{y}{2}=2$. $y=4$. But $y+2z=8$; then $4+2z=8$. $2z=8-4=4$. $z=2$. In the 1st, $x+y+z=12$. Substituting, $x+4+2=12$. Trans. $x=12-4-2=6$.

Problem 18.

1st. $x+y=a.$
2d. $x+z=b.$
3d. $y+z=c.$

Subtracting the 2d from the 1st, $y-z=a-b$. Adding this to the 3d, then $2y=a-b+c$. $y=\frac{a-b+c}{2}$. Subtracting the 3d from the 2d, then $x-y=b-c$. Adding this to the 1st, $2x=a+b-c$. $x=\frac{a+b-c}{2}$. Again, subtracting the 1st from the 2d, $z-y=b-a$. Adding this to the 3d, $2z=b+c-a$; and $z=\frac{b+c-a}{2}$.

Problem 19.

Let $x=$ A's money; $y=$ B's; $z=$ C's. By the supposition,

$$\left.\begin{array}{ll}\text{1st.} & x+\frac{y}{2}=100. \\ \text{2d.} & y+\frac{z}{3}=100. \\ \text{3d.} & z+\frac{x}{4}=100.\end{array}\right\} \text{ or, } \left\{\begin{array}{ll}\text{1st.} & 2x+y=200. \\ \text{2d.} & 3y+z=300. \\ \text{3d.} & 4z+x=400.\end{array}\right.$$

Mult. the 1st by 3, then $6x+3y=600$. Subtracting from this equation the 2d, then $6x-z=300$. Mult. by 4, $24x-4z=1200$. Adding the 3d to it, $25x=1600$. $x=64$. Substituting the value of x in the 1st, $128+y=200$. $y=72$. Substituting also in the 3d, $4z+64=400$. $4z=400-64=336$. Div. by 4, then $z=84$.

Problem 20.

Let $x=$ A's travel; $y=$ B's; $z=$ C's. By the conditions,

$$\left.\begin{array}{ll}\text{1st.} & x=4z+2y. \\ \text{2d.} & 2x+3y=17z. \\ \text{3d.} & x+y+z=62.\end{array}\right\}$$

Substituting, in the last two equations, the value of x as given in the 1st, we have in the 2d, $8z+4y+3y=17z$; and in the 3d, $4z+2y+y+z=62$. Red. the 2d, $7y=9z$; and $y=\frac{9z}{7}$. Red. the 3d, $3y+5z=62$. Substituting the value of y, then $\frac{27z}{7}+5z=62$. Clearing of fractions, $27z+35z=434$. $62z=434$. $z=7$. And $y=\frac{9z}{7}=9$. In the 1st, $x=4z+2y=28+18=46$.

Problem 21.

$$\left.\begin{array}{ll}\text{1st.} & \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62. \\ \text{2d.} & \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47. \\ \text{3d.} & \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.\end{array}\right\} \begin{array}{c}\text{By} \\ \text{clearing} \\ \text{of frac-} \\ \text{tions, we} \\ \text{have,}\end{array} \left\{\begin{array}{ll}\text{1st.} & 6x+4y+3z=744. \\ \text{2d.} & 20x+15y+12z=2820. \\ \text{3d.} & 15x+12y+10z=2280.\end{array}\right.$$

Mult. the 1st by 4, we have, $24x+16y+12z=2976$. Subtract the 2d from this, and $4x+y=156$. Mult. by 3, we have,

$12x+3y=468$. Mult. the 2d by 5, and $100x+75y+60z=14100$. Mult. the 3d by 6, and $90x+72y+60z=13680$. Subtract this from the former, and $10x+3y=420$. Take this from the equation found above [$12x+3y=468$], then $2x=48$. $x=24$. Substitute the value of x in the equation, $10x+3y=420$; and $240+3y=420$. $3y=420-240=180$; and $y=60$. Again, substitute the values of x and y in the 1st equation, viz. $6x+4y+3z=744$; then $144+240+3z=744$. Trans. $3z=744-144-240=360$. $z=120$.

Problem 22.

1st. $xy=600$.
2d. $xz=300$.
3d. $yz=200$.

Div. 1st by 2d, $\frac{y}{z}=2$; and $y=2z$. Substituting this value of y in the 3d, $2z^2=200$. $z^2=100$. $z=10$. $y=2z=20$. In the 2d, $xz=300$. But $z=10$; then $10x=300$. $x=30$.

Problem 23.

1st. $\frac{y}{2}+z+\frac{w}{2}=8$.
2d. $x+y+w=9$.
3d. $x+y+z=12$.
4th. $x+w+z=10$.

Clearing the 1st of fractions, we have,
(a) $y+2z+w=16$.
(b) $z-w=3$, by subst. 2d from 3d.
(c) $y-w=2$, by subst. 4th from 3d.

By adding (a) and (b) together, we have, $y+3z=19$. By adding (a) and (c), $2y+2z=18$; and $y+z=9$. Subtracting this from the equation [$y+3z=19$], $2z=10$. $z=5$. But $y+z=9$; then $y+5=9$. $y=4$. Again, by 3d, $x+y+z=12$; then $x+4+5=12$. $x=12-4-5=3$. In the 2d, $x+y+w=9$; then $3+4+w=9$. $w=9-3-4=2$. Thus $x=3$. $y=4$. $z=5$. $w=2$.

Problem 24.

1st. $w+50=x$.
2d. $x+120=3y$.
3d. $y+120=2z$.
4th. $z+195=3w$.

In 2d subst. the value of x as given in 1st, and we have 3 equations of 3 unknowns:

(a) $w+170=3y$.
(b) $y+120=2z$.
(c) $z+195=3w$.

Mult. (a) by 3, then $3w+510=9y$. Subtract (c) from it, and

$510=9y-z-195$. Trans. $z=9y-510-195=9y-705$. Div. (b) by 2, then $z=\frac{y+120}{2}$. Making the two values of z equal, $\frac{y+120}{2}=9y-705$. Mult. by 2, then $18y-1410=y+120$. Trans. $18y-y=1410+120$. Red. $17y=1530$. $y=90$. $z=\frac{y+120}{2}=\frac{90+120}{2}=105$. In 2d, $x+120=3y=270$. $x=270-120=150$. In 1st, $w+50=x=150$. $w=100$.

Problem 25.

Let x = left-hand digit; and y = right-hand digit. The local value of the left-hand digit is $10x$. By the conditions,

1st. $x=3y$.
2d. $10x+y-12=x^2$.

Squaring the 1st, $x^2=9y^2$. Substitute the value of x in the 2d, then $30y+y-12=9y^2$. Trans. and changing signs, $9y^2-31y=-12$. Completing the square by the second method, $324y^2-\pi+(31)^2=529$. Extracting the root, $18y-31=\sqrt{529}=23$. $18y=23+31=54$. $y=3$. $x=3y=9$. The number $=93$.

Problem 26.

Let x and y = the two digits. $10x+y$ = the number.

1st. $\frac{10x+y}{xy}=2$.
2d. $10x+y+27=10y+x$.

Clearing the 1st of fractions, $10x+y=2xy$. Trans. we have, $10x=2xy-y=y(2x-1)$. Div. by $2x-1$, $\frac{10x}{2x-1}=y$. Trans. in the 2d, then $10x-x+27=10y-y$. Red. $9x+27=9y$. Div. by 9, then $x+3=y=\frac{10x}{2x-1}$. Clearing of fractions, $2x^2-x+6x-3=10x$. Red. $2x^2-5x=3$. Completing the square by the second method, $16x^2-\pi+(5)^2=(5)^2+24=49$. Extracting the square root, $4x-5=\sqrt{49}=7$. $4x=7+5=12$. $x=3$. $y=x+3=6$. The number $=36$.

PROBLEM 27.

Let x and y = the two numbers. By the conditions,

1st. $3x-y=35$.
2d. $\frac{4x}{3y+1}=y$.

Trans. y in the 1st, then $3x=35+y$. Div. by 3, then $x=\frac{35+y}{3}$. Clearing the 2d of fractions, $4x=3y^2+y$. Div. by 4, then $x=\frac{3y^2+y}{4}$. Making the two values of x equal, $\frac{3y^2+y}{4}=\frac{35+y}{3}$. Clearing of fractions, $9y^2+3y=140+4y$. Trans. and red. $9y^2-y=140$. Completing the square by the second method, $324y^2-\pi+1=5040+1=5041$. Extracting the square root, $18y-1=\sqrt{5041}=71$. $18y=72$; and $y=4$. But $x=\frac{35+y}{3}$; then $x=\frac{35+4}{3}=\frac{39}{3}=13$.

PROBLEM 28.

Let x = the numerator; y = the denominator. By the conditions,

1st. $\frac{x+3}{y}=\frac{1}{3}$.
2d. $\frac{x}{y-1}=\frac{1}{5}$.

Clearing both of fractions, 1st. $3x+9=y$. 2d. $5x=y-1$. Trans. 1 in the 2d, then $5x+1=y$. But by the 1st, $y=3x+9=5x+1$. Trans. $9-1=5x-3x$. Red. $8=2x$. $x=4$. And $y=3x+9=12+9=21$. The fraction is $\frac{4}{21}$.

PROBLEM 29.

Let x = the value of the first horse; y = the value of the second.

1st. $x+10=2y$.
2d. $y+10=x-13$.

Trans. in the 1st, $x=2y-10$. Trans. in the 2d, $y+23=x$. Then $2y-10=y+23$. Trans. $2y-y=23+10$. Red. $y=33$. But $x=y+23=33+23=56$. 56 and 33 are the two values.

PROBLEM 30.

Let x = first part; y = second; z = third; then $90-x-y-z$ = fourth.

1st. $x+2=y-2$.
2d. $x+2=2z$.
3d. $2z=\frac{90-x-y-z}{2}$.

By the 1st, $x+4=y$. By the 2d, $\frac{x+2}{2}=z$. Substitute in the 3d the values of y and z, and we have,

$x+2=\frac{90-x-x-4-\frac{x+2}{2}}{2}$. Clearing of fractions, $2x+4=90-x-x-4-\frac{x+2}{2}$. Red. and trans. $4x=82-\frac{x+2}{2}$. Clearing again of fractions, $8x=164-x-2$. Trans. and red. $9x=162$; and $x=18$=first part. $y=x+4=18+4=22$=second. $z=\frac{x+2}{2}=\frac{18+2}{2}=\frac{20}{2}=10$=third. $90-x-y-z=90-18-22-10$. Red. $90-x-y-z=40$=fourth part. The parts are, 18, 22, 10, 40.

Problem 31.

Let x, y, and z = the numbers. By the supposition,

1st. $x+\frac{y+z}{2}=120$.
2d. $y+\frac{z-x}{5}=70$.
3d. $\frac{x+y+z}{2}=95$.

By clearing of fractions, we have,

1st. $2x+y+z=240$.
2d. $5y+z-x=350$.
3d. $x+y+z=190$.

Subtracting the 3d from the 1st, we have $x=50$. Subtracting the 3d from the 2d, and $4y-2x=160$. But $x=50$; and $2x=100$. Then $4y-100=160$. $4y=160+100=260$. $y=65$. $x+y+z=190$; then $50+65+z=190$. $z=190-50-65=75$.

Problem 32.

Let x and y = the two numbers. By the conditions,

1st. $x-y:x+y::2:3$.
2d. $x+y:xy::3:5$.

Changing to the form of an equation,
1st. $3x-3y=2x+2y$.
2d. $5x+5y=3xy$.

Trans. in the 1st, $3x-2x=2y+3y$. Red. $x=5y$. Trans.

$5x$ in the 2d, then $5y=3xy-5x=x(3y-5)$. Div. by $(3y-5)$, then $\frac{5y}{3y-5}=x=5y$. Div. by $5y$, we have $\frac{1}{3y-5}=1$. Clearing of fractions, $1=3y-5$. Trans. $6=3y$. $2=y$. But $x=5y=5\times 2=10$. 10 and 2 are the numbers.

Problem 33.

Let x = the price per dozen of the sherry; and y = that of the port.

$$\left.\begin{array}{l}\text{1st. } 20y+30x=120.\\ \text{2d. } 30y+25x=140.\end{array}\right\}\begin{array}{l}\text{Mult. by 3 the 1st, } 60y+90x=360.\\ \text{Mult. by 2 the 2d, } 60y+50x=280.\end{array}$$

Subtracting the 2d from the 1st, then $40x=80$; and $x=2$. But $20y+30x=120$: as $30x=60$, then $20y+60=120$. Trans. $20y=120-60=60$. $y=3$. The price of the sherry per dozen is two guineas; and of the port, three guineas.

Problem 34.

Let x = number of gallons of brandy; y = number of gallons of water.

$$\left.\begin{array}{l}\text{1st. } x+6:y+6::7:6.\\ \text{2d. } x-6:y-6::6:5.\end{array}\right\}\begin{array}{l}\text{Chang. to}\\ \text{equations,}\end{array}\left\{\begin{array}{l}\text{1st. } 6x+36=7y+42.\\ \text{2d. } 5x-30=6y-36.\end{array}\right.$$

Mult. the 1st by 5, then $30x+180=35y+210$. Mult. the 2d by 6, then $30x-180=36y-216$. Subt. the 2d from the 1st, we have $360=-y+426$. Trans. $y=426-360=66$ = number of gallons of water. Trans. in the 1st, $6x=42-36+7y=6+7y$. But $7y=462$. Then $6x=6+462=468$. $x=78$ = number of gallons of brandy.

Problem 35.

Let x = the numerator; y = the denominator. By the supposition,

$$\left.\begin{array}{l}\text{1st. } \frac{2x}{y+7}=\frac{2}{3}.\\ \text{2d. } \frac{x+2}{2y}=\frac{3}{5}.\end{array}\right\}\begin{array}{l}\text{Clearing both of frac-}\\ \text{tions, we have}\end{array}\left\{\begin{array}{l}\text{1st. } 6x=2y+14.\\ \text{2d. } 5x+10=6y.\end{array}\right.$$

Mult. the 1st by 5, then $30x=10y+70$. Mult. the 2d by 6, and we have $30x+60=36y$. Subt. the former from the latter, $60=26y-70$. Trans. $26y=60+70=130$. $y=5=$ denominator. In the 1st, $6x=2y+14$. But $2y=10$; then $6x=10+14=24$. $x=4=$ numerator. The fraction is $\frac{4}{5}$.

Problem 36.

Let $x=$ the number of apples; $y=$ number of pears. Then $\frac{x}{4}=$ cost of the apples, and $\frac{y}{5}=$ cost of the pears. By the conditions,

1st. $\frac{x}{4}+\frac{y}{5}=30.$
2d. $\frac{x}{8}+\frac{y}{15}=13.$
Clearing both of fractions,
1st. $5x+4y=600.$
2d. $15x+8y=1560.$

Mult. the 1st by 3, then $15x+12y=1800$. Subtract from this equation the 2d, and $4y=240$. $y=60=$ number of pears. In the 1st, $5x+4y=600$. But $4y=240$; then $5x+240=600$. $5x=600-240=360$. $x=72=$ number of apples.

PROPORTION.

Example 1.

Let $x=$ the greater part; $49-x=$ less. By the conditions, $x+6:38-x::9:2$. Adding antecedents to consequents, $x+6:44::9:11$. Div. consequents by 11, then $x+6:4::9:1$. Changing to an equation, $x+6=36$. $x=36-6=30$. $49-x=49-30=19$. 30 and 19 are the two parts.

Example 2.

Let $x=$ the number; then $x+1:x+5::x+5:x+13$. Subt. antecedents from consequents, $x+1:4::x+5:8$ (Art. 389. 4). Subt. the 1st ratio from the last, $x+1:4::4:4$. Div. consequents by 4, then $x+1:1::4:1$. Changing to an equation, $x+1=4$. $x=4-1=3$.

EXAMPLE 3.

Let x and y = the numbers. By the supposition,
$x:y::x+y:42.$ } Then, by Art. 384, $x+y:42::x-y:6.$
$x:y::x-y:6.$ } Inverting the means, $x+y:x-y::42:6.$
Div. then $x+y:x-y::7:1.$ Adding and subt. terms (Art. 389. 7), $2x:2y::8:6.$ Div. then $x:y::4:3$; and $3x=4y.$ $x=\frac{4y}{3}.$ Changing the second proportion to an equation, $6x=y(x-y).$ Substituting the value of x, then $8y=y\left(\frac{4y}{3}-y\right)=y\times\frac{y}{3}=\frac{y^2}{3}.$ Div. by y, then $8=\frac{y}{3}.$ $24=y.$ $x=\frac{4y}{3}=32.$

EXAMPLE 4.

Let x = greater part; $18-x$ = less. By the conditions, $x^2:(18-x)^2::25:16.$ Extracting the root, by Art. 391, $x:18-x::5:4.$ Adding antecedents to consequents, $x:18::5:9.$ Div. consequents by 9, then $x:2::5:1.$ Changing to an equation, $x=10.$ $18-x=8.$

EXAMPLE 5.

Let x = the greater part; $14-x$ = the less. By the conditions, $\frac{x}{14-x}:\frac{14-x}{x}::16:9.$ Mult. terms, $x^2:(14-x)^2::16:9.$ By Art. 391, extracting the root, $x:14-x::4:3.$ Adding antecedents to consequents, $x:14::4:7.$ Div. consequents by 7, then $x:2::4:1.$ Changing to an equation, $x=8.$ $14-x=6.$

EXAMPLE 6.

Let x = the greater part; $20-x$ = the less. By the supposition, $x:20-x::9:1.$ Adding antecedents to consequents, $x:20::9:10.$ Div. consequents by 10, then $x:2::9:1.$ Changing to an equation, $x=18.$ $20-x=2.$ Let w = the mean proportional, then, by Art. 376, $w^2=18\times2=36.$ $w=\sqrt{36}=6.$

Example 7.

Let x and y = the two numbers. By the conditions, $x^3-y^3:(x-y)^3::19:1$. Expanding the second term, $x^3-y^3:x^3-3x^2y+3xy^2-y^3::19:1$. Subtracting consequents from antecedents, $3x^2y-3xy^2:(x-y)^3::18:1$. Div. first couplet by $x-y$ (Art. 382. 5), $3xy:(x-y)^2::18:1$. But, by the supposition, $xy=24$; then $3xy=72$. Substituting this, $72:(x-y)^2::18:1$. Div. antecedents by 18, $4:(x-y)^2::1:1$. $(x-y)^2=4$. Extracting root, $x-y=2$. $x=2+y$. But $xy=24$; and $x=\frac{24}{y}=2+y$. Clearing of fractions, $y^2+2y=24$. Completing the square, $y^2+2y+1=25$. Extracting the root, $y+1=\sqrt{25}=5$. $y=5-1=4$. $x=y+2=4+2=6$.

Example 8.

$(a+x)^2:(a-x)^2::x+y:x-y$. Expanding the first couplet, $a^2+2ax+x^2:a^2-2ax+x^2::x+y:x-y$. Adding and subt. terms (Art. 389. 7), $2a^2+2x^2:4ax::2x:2y$. Div. by 2, then $a^2+x^2:2ax::x:y$. Transferring the factor x, $a^2+x^2:2a::x^2:y$. Inverting the means, $a^2+x^2:x^2::2a:y$. Subt. consequents from antecedents, $a^2:x^2::2a-y:y$. Extracting the root, $a:x::\sqrt{2a-y}:\sqrt{y}$.

Example 9.

1st. $a^3:b^3::x:y$.
2d. $a:b::\sqrt[3]{c+x}:\sqrt[3]{d+y}$.
} Involving the 2d to the third power, $a^3:b^3::c+x:d+y$. By equality of ratios (Art. 384), $c+x:d+y::x:y$. Inverting the means, $c+x:x::d+y:y$. Subtracting consequents from antecedents, $c:x::d:y$. Then $dx=cy$.

Example 10.

Let x and y = the numbers. Then $xy=135$; and $x^3-y^3:(x-y)^3::4:1$. Div. the first couplet by $x-y$, $x+y:x-y::4:1$. Adding and subtracting terms, $2x:2y::5:3$. Div.

by 2, $x:y::5:3$. $3x=5y$. $x=\frac{5y}{3}$. But $xy=135$, and $x=\frac{135}{y}=\frac{5y}{3}$. Clearing of fractions, $405=5y^2$. $81=y^2$. $y=\sqrt{81}=9$. $x=\frac{5y}{3}=15$.

Example 11.

Let x and y = the numbers. By the conditions,

1st. $x-y:x+y::2:3$.
2d. $x+y:xy::3:5$.

Adding and subtracting terms in the 1st, $2x:2y::5:1$. Div. by 2, then $x:y::5:1$. $x=5y$. Substituting this value of x in the 2d, $6y:5y^2::3:5$. Div. by y, then $6:5y::3:5$. Div. by 3, we have $2:5y::1:5$. Again, div. by 5, then $2:y::1:1$. $2=y$. $x=5y=10$.

Example 12.

Let x = the greater part; and $24-x$ = the less. Then $x(24-x):x^2+(24-x)^2::3:10$. Expanding, $24x-x^2:576-48x+2x^2::3:10$. Div. consequents by 2, $24x-x^2:288-24x+x^2::3:5$. Adding antecedents to consequents, $24x-x^2:288::3:8$. Div. consequents by 8, $24x-x^2:36::3:1$. Changing to an equation, $24x-x^2=108$. Changing signs, $x^2-24x=-108$. Completing the square, $x^2-24x+144=144-108=36$. Extracting the root, $x-12=\sqrt{36}=6$. $x=6+12=18$. $24-x=6$. Then 18 = the greater, and 6 = the less.

Example 13.

Let x = the quantity of rum, and y = the quantity of brandy.

1st. $x-y:y::100:x$.
2d. $x-y:x::4:y$.

Inverting the means in the 2d, $x-y:4::x:y$. Mult. consequents by 25, then $x-y:100::x:25y$. Inverting the means in the 1st, $x-y:100::y:x$. By equality of ratios, we have $x:25y::y:x$. Changing to an equation, $x^2=25y^2$. Evolving, $x=5y$. Substituting the value of x in the 1st, then $4y:y::100:5y$. Div. the 1st couplet by y and the last one by 5, then $4:1::20:y$. $4y=20$. $y=5$. $x=5y=25$.

EXAMPLE 14.

Let $2x$ and $3x$ = the numbers. By the conditions, $3x+6 : 2x-6 :: 3 : 1$. Mult. antecedents by 2, and consequents by 3, then $6x+12 : 6x-18 :: 6 : 3$. Subtracting consequents from antecedents, $30 : 6x-18 :: 3 : 3$. Then $6x-18=30$. $6x=30+18=48$. $x=8$. $2x=16$=the less number. $3x=24$=the greater number.

EXAMPLE 15.

Let x and y = the numbers. Then $xy=320$; and $x^3-y^3 : (x-y)^3 :: 61 : 1$. Expand. $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 61 : 1$. Subt. consequents from antecedents, $3x^2y-3xy^2 : (x-y)^3 :: 60 : 1$. Div. first couplet by $x-y$, then $3xy : (x-y)^2 :: 60 : 1$. Div. antecedents by 3, we have $xy : (x-y)^2 :: 20 : 1$. But $xy=320$; then $320 : (x-y)^2 :: 20 : 1$. Div. by 20, $16 : (x-y)^2 :: 1 : 1$. $16=(x-y)^2$. $4=x-y$, by evolution. $4+y=x=\frac{320}{y}$. Clearing of fractions, $y^2+4y=320$. Completing the square, $y^2+4y+4=320+4=324$. Extracting the root, $y+2=\sqrt{324}=18$. $y=18-2=16$. $x=y+4=16+4=20$.

EXAMPLE 16.

Let x and y = the two numbers. By the conditions,

1st. $x : y :: 4^2 : 3^2$, or $x : y :: 16 : 9$.
2d. $x : 24 :: 24 : y$.
} Inverting the means in the 1st, $x : 16 :: y : 9$.

Mult. the consequents by 3, and div. them by 2, then $x : 24 :: y : 13\frac{1}{2}$. By equality of ratios, $y : 13\frac{1}{2} :: 24 : y$. $y^2=324$. $y=\sqrt{324}=18$. Substituting this value in the 1st, $x : 18 :: 16 : 9$. Div. consequents by 9, we have $x : 2 :: 16 : 1$. $x=32$.

ARITHMETICAL PROGRESSION.

THE manner in which the primary formula $z=a+(n-1)\times d$ is obtained is so clearly explained in the Algebra, that we will not trouble the student to follow through another exposition of the subject; being satisfied that if he has carefully examined the former, he cannot but understand it. Let us take the equation $z=a+(n-1)\times d$ as already established, understanding that $z=$ the last term, $a=$ the first term, $n=$ number of terms, $d=$ the common difference; we will now proceed to find the values of the other quantities, viz. a, n, d. First, $z=a+(n-1)\times d$. Trans. $(n-1)\times d$, then $z-(n-1)\times d=$ $a=$ the first term. Again, taking the same equation and trans. a, then $z-a=(n-1)\times d$. Div. the equation by $(n-1)$, we have $\frac{z-a}{n-1}=d=$ the common difference. Thirdly, taking the equation as above, $z-a=(n-1)\times d$, and div. by d, then $\frac{z-a}{d}=n-1$. Trans. the 1, we have $\frac{z-a}{d}-1=n=$ number of terms. Thus the value of the four quantities is as follows: First term $=a=z-(n-1)\times d$. Common difference $=d=\frac{z-a}{n-1}$. Last term $=z=a+(n-1)\times d$. No. of terms $=n=\frac{z-a}{d}+1$.

PROBLEM 1.

$a=7$, $d=3$, $n=9$; to find z. By the formula, $z=a+(n-1)\times d$. Substituting the values of the several letters, $z=7+(9-1)\times 3=7+8\times 3=7+24=31$.

PROBLEM 2.

$z=60$, $n=12$, $d=5$; to find a. By the formula, $a=z-(n-1)\times d$. Substituting the numbers, $a=60-(12-1)\times 5=60-11\times 5=60-55=5$.

PROBLEM 3.

$a=1$, $z=43$, $n=8$; to find d. By the formula, $d=\frac{z-a}{n-1}$. Substituting the numerical values of a, z, and n, then $d=\frac{43-1}{8-1}=\frac{42}{7}=6$. Adding 6 to the first term, gives the second; and adding 6 to that, gives the third, &c. Consequently the series is, 1, 7, 13, 19, 25, 31, 37, 43.

In the summation of the series, we have the following formula, $s=\frac{a+z}{2}\times n$; the method of obtaining which is clearly explained in the Algebra. In this equation, the signification of the letters is the same as in the preceding formulas; the additional letter, s, denoting the *sum of the series*. It will be remembered, that in the foregoing article, $z=a+(n-1)\times d$. Substituting that value of z in this equation, we have $s=\frac{a+a+(n-1)\times d}{2}\times n=\frac{2a+(n-1)\times d}{2}\times n$. Clearing the equation of fractions, we have $2s=[2a+(n-1)\times d]\times n=2an+(n-1)\times dn$. Expanding the vinculated quantity, $(n-1)\times dn=dn^2-dn$. Substituting this, $2s=2an+dn^2-dn$. Transposing, $2s-dn^2+dn=2an$. Div. by $2n$, we find that $\frac{2s-dn^2+dn}{2n}=a=$ *the first term.*

Again, to find the value of d, taking the equation $2s=2an+dn^2-dn$, and trans. $2an$, then $2s-2an=dn^2-dn$. But $dn^2-dn=(n^2-n)\times d$. Then $2s-2an=(n^2-n)\times d$. Div. the equation by (n^2-n), we shall have this equation, $\frac{2s-2an}{n^2-n}=d=$ *the common difference.*

Thirdly, to find the value of n; let us take the same equation as above, viz. $2s=2an+dn^2-dn$. As $2an-dn=n(2a-d)$; then $2s=dn^2+n(2a-d)$. Dividing by d, we have $\frac{2s}{d}=$

$n^2+n\left(\frac{2a-d}{d}\right)$. Here it will be observed, that we have an affected quadratic equation; for in the second member of the equation there is the *square* of a certain quantity (n) and *the first power* of the same quantity with its coefficient $\left(\frac{2a-d}{d}\right)$. Completing the square according to the second method, we have the following: $\frac{8s}{d}+\frac{(2a-d)^2}{d^2}=4n^2+\pi+\frac{(2a-d)^2}{d^2}$. Reducing the first member of the equation to the common denominator of d^2, then $\frac{8s}{d}+\frac{(2a-d)^2}{d^2}=\frac{8ds+(2a-d)^2}{d^2}$. Substituting this, $\frac{8ds+(2a-d)^2}{d^2}=4n^2+\pi+\frac{(2a-d)^2}{d^2}$. Extracting the square root, $\frac{\sqrt{8ds+(2a-d)^2}}{d}=2n+\frac{2a-d}{d}$. Trans. $\frac{2a-d}{d}$, then $\frac{\sqrt{8ds+(2a-d)^2}-2a+d}{d}=2n$. Div. by 2, then $\frac{\sqrt{8ds+(2a-d)^2}-2a+d}{2d}=n=$ *the number of terms.*

EXAMPLES UNDER THIS HEAD.

EXAMPLE 1.

$a=3$; $d=2$; $n=20$. By the formula, $s=\frac{2a+(n-1)\times d}{2}\times n$.

Then $s=\frac{2\times 3+(20-1)\times 2}{2}\times 20=\frac{6+38}{2}\times 20=440$.

EXAMPLE 2.

The first stone being 1 yard from the box, to go to the stone and bring it to the box would give a distance of 2 yards; consequently $2=$ the first term. The second stone being 2 yards from the box, to it and back to the box would give a distance of 4 yards, or the second term. The difference between this and the first term being 2 yards, $2=$ the common difference. Then $a=2$; $d=2$; $n=100$. But $s=\frac{2a+(n-1)\times d}{2}\times n$. Substituting the numerical values of the letters, $s=\frac{2\times 2+(100-1)\times 2}{2}\times 100$. Red. $s=\frac{4+198}{2}\times 100=10100$. 10100 yards = 5 miles and 1300 yards.

EXAMPLE 3.

$a=\frac{1}{3}$; $d=\frac{1}{3}$; $n=150$. By the same formula,
$s=\frac{\frac{1}{3}\times 2+(150-1)\times\frac{1}{3}}{2}\times 150$. Then $s=\frac{\frac{2}{3}+149\times\frac{1}{3}}{2}\times 150$
$=\frac{\frac{2}{3}+\frac{149}{3}}{2}\times 150=\frac{\frac{151}{3}}{2}\times 150=3775$.

EXAMPLE 4.

$a=5$; $n=30$; $s=1455$. By the formula, $d=\frac{2s-2an}{n^2-n}$.
Substituting, $d=\frac{2\times 1455-2\times 5\times 30}{(30)^2-30}=\frac{2910-300}{900-30}=\frac{2610}{870}$
$=3$.

EXAMPLE 5.

$a=7$; $d=2$; $s=567$. But $n=\frac{\sqrt{8ds+(2a-d)^2}-2a+d}{2d}$. Then n
$=\frac{\sqrt{8\times 2\times 567+(14-2)^2}-2\times 7+2}{2\times 2}=\frac{\sqrt{9072+144}-14+2}{4}$.
Continuing the reduction, $n=\frac{\sqrt{9216}-12}{4}$. $\sqrt{9216}=96$.
Then $n=\frac{96-12}{4}=21$.

EXAMPLE 6.

$a=1$; $d=\frac{1}{2}$; $n=32$. By the formula, $s=\frac{2a+(n-1)\times d}{2}\times n$.
Then $s=\frac{2+(32-1)\times\frac{1}{2}}{2}\times 32=\frac{2+31\times\frac{1}{2}}{2}\times 32=\frac{35}{4}\times 32=$
280.

Example 7.

$a=10$; $d=20$; $n=47$. By the formula,

$$s=\frac{2\times10+(47-1)\times20}{2}\times47. \quad \text{Red.}\ s=\frac{20+46\times20}{2}\times47$$

$$=\frac{20+920}{2}\times47=470\times47=22090=\$220.90.$$

Example 8.

$$a=1;\ d=1;\ n=365;\ s=\frac{2a+(n-1)\times d}{2}\times n=\frac{2+(365-1)\times1}{2}$$

$\times365$. Red. $s=\dfrac{2+364}{2}\times365=183\times365=66795$ cts. $=$ \$667.95.

PROBLEMS UNDER THE SAME HEAD.

Problem 1.

Let $x=$ the second number; and $y=$ common difference. Then the series will be, $x-y$, x, $x+y$, $x+2y$. By the conditions of the problem,

1st. $x-y+x+x+y+x+2y=56$; or reducing, $4x+2y=56$. $2x+y=28$.

2d. $(x-y)^2+x^2+(x+y)^2+(x+2y)^2=864$. Expanding, we have $x^2-2xy+y^2+x^2+x^2+2xy+y^2+x^2+4xy+4y^2=864$. Reducing the equation, $4x^2+4xy+6y^2=864$. But by the 1st, $2x+y=28$. Squaring this equation, we have $4x^2+4xy+y^2=784$. Then we have these two equations,

$$\left.\begin{array}{l}4x^2+4xy+6y^2=864.\\4x^2+4xy+\ \ y^2=784.\end{array}\right\}$$ Subtracting the lower equation from the upper one,

$5y^2=80$. $y^2=16$. Extracting the root, $y=4$.

By the 1st, $2x+y=28$. $2x=28-y=28-4=24$. $x=12$.
The series will be, $12-4$, 12, $12+4$, $12+8$.
Or, 8, 12, 16, 20.

PROBLEM 2.

Let $x=$ the second number; and $y=$ the common difference. Then the series will be, $x-y$, x, $x+y$. By the conditions of the problem,

1st. $x-y+x+x+y=9$. And red. $3x=9$. $x=3$.
2d. $(x-y)^3+x^3+(x+y)^3=153$. Expanding the first member of the equation, $x^3-3x^2y+3xy^2-y^3+x^3+x^3+3x^2y+3xy^2+y^3=153$. Red. $3x^3+6xy^2=153$. Div. by 3, then $x^3+2xy^2=51$. But $x=3$, then $x^3=27$; and $2x=6$. Substituting, $27+6y^2=51$. Trans. $6y^2=51-27=24$. $y^2=4$. $y=2$. The series will be, $3-2$, 3, $3+2$; or, 1, 3, 5.

PROBLEM 3.

Let $x=$ the second number; and $y=$ the common difference. The series will be, $x-y$, x, $x+y$. By the conditions of the problem,

1st. $x-y+x+x+y=15$. Red. $3x=15$; and $x=5$.
2d. $(x-y)^2+(x+y)^2=58$. Exp. $x^2-2xy+y^2+x^2+2xy+y^2=58$. Red. $2x^2+2y^2=58$. Div. by 2, then $x^2+y^2=29$. But $x=5$, and $x^2=25$; then $25+y^2=29-y^2=29-25=4$. $y=2$. The series will be, $5-2$, 5, $5+2$; or, 3, 5, 7.

PROBLEM 4.

Let the series be, $x-3y$, $x-y$, $x+y$, $x+3y$. By the conditions of the problem,

1st. $(x-3y)^2+(x-y)^2=34$; and
2d. $(x+y)^2+(x+3y)^2=130$. } Expanding both of these equations,

$x^2-6xy+9y^2+x^2-2xy+y^2=34$; or, $2x^2-8xy+10y^2=34$.
$x^2+2xy+y^2+x^2+6xy+9y^2=130$; or, $2x^2+8xy+10y^2=130$.

Subt. the upper equation from the lower, $16xy=96$.

Div. by $16x$, then $y=\frac{96}{16x}=\frac{6}{x}$. Taking the same two equations and adding them, the two $8xy$ cancel, and we have $4x^2+20y^2=164$. Div. by 4, then $x^2+5y^2=41$. Trans. $5y^2=$

$41-x^2$. Div. $y^2=\frac{41-x^2}{5}$. Extracting the root, $y=\sqrt{\frac{41-x^2}{5}}$.

But $y=\frac{6}{x}$; then $\sqrt{\frac{41-x^2}{5}}=\frac{6}{x}$. Involving both sides, $\frac{41-x^2}{5}$ $=\frac{36}{x^2}$. Clearing of fractions, $41x^2-x^4=180$. Changing the signs, $x^4-41x^2=-180$. Completing the square by the second method, $4x^4-x+(41)^2=(41)^2-720=961$. Extracting the root, $2x^2-41=\sqrt{961}=31$. Trans. $2x^2=31+41=72$. $x^2=36$. Evolving, $x=\sqrt{36}=6$. But $y=\frac{6}{x}=\frac{6}{6}=1$. Then the series will be, $6-3$, $6-1$, $6+1$, $6+3$; or, 3, 5, 7, 9.

Problem 5.

Let the series $x-y$, x, $x+y$, represent the three digits. Then the number $=100(x-y)+10x+(x-y)$; which reduced, equals $111x-99y$. Then

1st. $\frac{111x-99y}{3x}=26$. 2d. $\left\{\begin{array}{l}111x-99y+198=\\100(x+y)+10x+(x-y).\end{array}\right.$

Clearing the 1st of fractions, $111x-99y=78x$. Transposing, $111x-78x=99y$. Red. $33x=99y$; and $x=3y$. Expanding the 2d, we have $111x-99y+198=100x+100y+10x+x-y$. Trans. $198=100x+10x+x-111x+100y+99y-y$. Red. $198=198y$. $1=y$. But $x=3y$; then $x=3$. The digits will be, $3-1$, 3, $3+1$; or, 2, 3, 4; and the number is 234.

Problem 6.

Let the series be, $x-3y$, $x-y$, $x+y$, $x+3y$. By the conditions,

1st. $\left\{\begin{array}{l}(x-3y)^2+(x+3y)^2=200;\text{ or, }x^2-6xy+9y^2+x^2+6xy+9y^2\\=200.\end{array}\right.$

2d. $\left\{\begin{array}{l}(x-y)^2+(x+y)^2=136;\text{ or, }x^2-2xy+y^2+x^2+2xy+y^2\\=136.\end{array}\right.$

Reducing the 1st, $2x^2+18y^2=200.$ } Subtracting the upper
" " 2d, $2x^2+\ \ 2y^2=136.$ } equation from the lower

one, we have, $16y^2=64$. Div. by 16, then $y^2=4$.

Evolving, $y=2$. In the 1st, substituting for $18y^2$ its value 72, then we have $2x^2+72=200$. Trans. $2x^2=200-72=128$. $x^2=64$. Evolving, $x=\sqrt{64}=8$. The series is, $8-6$, $8-2$, $8+2$, $8+6$; or, 2, 6, 10, 14.

Problem 7.

Let the series be, $x-3y$, $x-y$, $x+y$, $x+3y$. By the conditions,

$x-3y+x-y+x+y+x+3y=28$. Red. $4x=28$. $x=7$. $(x-3y)\times(x-y)\times(x+y)\times(x+3y)=585$. Expanded, $x^4-10x^2y^2+9y^4=585$. But by the 1st, $x=7$; then $x^4=2401$, and $10x^2=490$. Substituting these values in the last equation, $2401-490y^2+9y^4=585$. Trans. $9y^4-490y^2=585-2401=-1816$. Completing the square by the second method, $324y^4-\pi+(490)^2=(490)^2-1816\times36=174724$. Extracting the root, $18y^2-490=\sqrt{174724}=\pm418$. Trans. $18y^2=490\pm418=908$, or 72. As the last value, which is obtained by using the minus sign, is the only reducible one, we shall employ that one. $18y^2=72$. Div. by 18, then $y^2=4$. Evolving, $y=2$. Then the series is, $7-6$, $7-2$, $7+2$, $7+6$; or, 1, 5, 9, 13.

GEOMETRICAL PROGRESSION.

In Geometrical Progression, the four formulas, or the values of a, z, n, and r, are obtained in a way similar to those of Arithmetical Progression. Premising that the student has examined the process in the book, we will proceed to the examples.

Problem I.

$a=4$; $z=256$; $n=4$: to find r. By the formula, $r=\left(\frac{z}{a}\right)^{\frac{1}{n-1}}$

Substituting, $r=\left(\frac{256}{4}\right)^{\frac{1}{4-1}}=(64)^{\frac{1}{3}}$ or $\sqrt[3]{64}=4$. 4 being the ratio, then the two means are, 16 and 64; and the series $=$ 4, 16, 64, 256.

PROBLEM 2.

$a=\frac{1}{9}$; $z=9$; $n=5$: to find r. But $r=\left(\frac{z}{a}\right)^{\frac{1}{n-1}}=\left(\frac{9}{\frac{1}{9}}\right)^{\frac{1}{5-1}}$ $=(81)^{\frac{1}{4}}=3$. Then the three means are, $\frac{1}{3}$, 1, 3; and the series $=\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9.

Let $s=$ sum of the terms; and let the series be, a, ar, ar^2, ar^3, $\ldots ar^{n-1}$; then $s=a, ar+ar^2+ar^3, \ldots +ar^{n-1}$. Mult. the series by r, $rs=\ ar+ar^2+ar^3, \ldots +ar^{n-1}+ar^n$. Subt. 1st from 2d, $rs-s=-a+ar^n=ar^n-a$. Div. by $(r-1)$, then $s=\frac{ar^n-a}{r-1}$. But ar^n is the last term, and equals rz; then $s=\frac{rz-a}{r-1}$.

PROBLEM 1.

$a=6$; $z=1458$; $r=3$. But $s=\frac{rz-a}{r-1}=\frac{3\times 1458-6}{3-1}=\frac{4368}{2}$ $=2184$.

PROBLEM 2.

$a=\frac{1}{2}$; $r=\frac{1}{3}$; $n=5$; then $ar^n=\frac{1}{2}\times(\frac{1}{3})^5=\frac{1}{2}\times\frac{1}{243}=\frac{1}{486}$. Using the formula $s=\frac{ar^n-a}{r-1}$, we have $s=\frac{\frac{1}{486}-\frac{1}{2}}{\frac{1}{3}-1}=\frac{-\frac{242}{486}}{-\frac{2}{3}}$ $=\frac{242\times 3}{486\times 2}=\frac{121}{162}$.

PROBLEM 3.

$a=1$; $r=3$; $n=12$; then $ar^n=1\times 3^{12}=1\times 531441$. By the same formula, $s=\frac{ar^n-a}{r-1}=\frac{531441-1}{3-1}=\frac{531440}{2}=$ 265720.

PROBLEM 4.

$a=1$; $r=\frac{2}{3}$; $n=10$; then $ar^n=1\times(\frac{2}{3})^{10}=\frac{1024}{59049}$. Using the same formula, $s=\frac{ar^n-a}{r-1}=\frac{\frac{1024}{59049}-1}{\frac{2}{3}-1}=\frac{174075}{59049}$.

PROBLEM 1.

Let x, y, and z = the three numbers. By the conditions,

1st. $x:y::y:z$, or $xz=y^2$.
2d. $x+y+z=14$.
3d. $x^2+y^2+z^2=84$.

Substituting the value of y^2 in the 3d as found in the 1st, then $x^2+xz+z^2=84$. As xz is equal to y^2, add xz to the 1st member, and y^2 to the 2d, $x^2+2xz+z^2=84+y^2$. Extracting the square root, $x+z=\sqrt{84+y^2}$. Trans. y in the 2d, then $x+z=14-y$. Making the two values of $(x+z)$ equal, $14-y=\sqrt{84+y^2}$. Involving to the second power, $196-28y+y^2=84+y^2$. Transposing, $28y=-84+196=112$. $4=y$. But $x+z=14-y=14-4=10$. Transposing, $x=10-z$. Again, $xz=y^2=16$. $x=\frac{16}{z}$. Then $10-z=\frac{16}{z}$. Clearing of fractions, $10z-z^2=16$. Changing the signs, $z^2-10z=-16$. Completing the square, $z^2-10z+25=25-16=9$. Extracting the root, $z-5=3$. $z=5+3=8$. $x=10-z=10-8=2$. The series is, 2, 4, 8.

PROBLEM 2.

Let x, xy, xy^2 = the series. By the supposition,

1st. $x^2y^2=64$.
2d. $x^2+x^2y^2+x^2y^6=584$.

Subt. the 1st from the 2d, we have $x^2+x^2y^6=520$. Div. by $(1+y^6)$, $x^2=\frac{520}{1+y^6}$. Div. the 1st by y^2, then $x^2=\frac{64}{y^2}=\frac{520}{1+y^6}$. Clearing of fractions, $64+64y^6=520y^2$. Div. by 8, we have $8+8y^6=65y^2$. Trans. $8y^6-65y^2=-8$. Completing the square by the second method, $256y^6-*+(65)^2=(65)^2-256=3969$. Extracting the root, $16y^3-65=\sqrt{3969}=63$. $16y^3$

$=63+65=128$. $y^3=8$. $y=2$. But $x^3y^3=64$; then $8x^3=64$. $x^3=8$. $x=2$. The series is, 2, 4, 8.

Problem 3.

Let x, y, z = the three numbers. By the conditions,

1st. $x:y::y:z$, or $xz=y^2$.
2d. $x+z=52$.
3d. $y^2=100$.

$xz=y^2=100$. $x=\frac{100}{z}$. Trans. in the 2d, $x=52-z$. Making the two values of x equal, $52-z=\frac{100}{z}$. Clearing of fractions, $52z-z^2=100$. Changing signs, $z^2-52z=-100$. Completing the square, $z^2-52z+(26)^2=(26)^2-100=576$. Extracting the root, $z-26=\sqrt{576}=24$. $z=24+26=50$. But $x=52-z=52-50=2$. As $y^2=100$, then $y=10$. The series is, 2, 10, 50.

Problem 4.

Let x, xy, xy^2, xy^3 = the series. Then, by the conditions,

1st. $x+xy=15$.
2d. $xy^2+xy^3=60$.

Div. the 2d by the 1st, we have $y^2=4$. $y=2$. Substituting in the 1st, then $x+2x=15$. $3x=15$. $x=5$. The series is, 5, 10, 20, 40.

Problem 5.

Let x, xy, xy^2 = the series. By the supposition,

1st. $x+xy+xy^2=210$.
2d. $xy^2=x+90$.

Substituting, in the 1st, the value of xy^2, $x+xy+x+90=210$. Trans. and red. $2x+xy=120$. Div. by $(2+y)$, then $x=\frac{120}{2+y}$. Trans. x in the 2d, $xy^2-x=90$. Div. by (y^2-1), then $x=\frac{90}{y^2-1}$. Making the two values of x equal, $\frac{90}{y^2-1}=\frac{120}{2+y}$. Clearing of fractions, $180+90y=120y^2-120$. Div. by 30, then $6+3y=4y^2-4$. Trans. $4y^2-3y=6+4=10$. Completing the square by the second method, $64y^2-\pi+9=160+9=169$. Extracting the root, $8y-3=\sqrt{169}=13$. $8y$

$=16$. $y=2$. But $x=\frac{120}{y+2}=\frac{120}{4}=30$. The portions of the servants are, \$30, \$60, \$120.

PROBLEM 6.

Let x, xy, $xy^2=$ the series. By the conditions, 1st. $xy^2=x+15$. 2d. $x^2y^4-x^2 : x^2+x^2y^2+x^2y^4 :: 5 : 7$. Div. the 1st couplet by x^2, then $y^4-1 : 1+y^2+y^4 :: 5 : 7$. Subt. antecedents from consequents, $y^4-1 : y^2+2 :: 5 : 2$. Changing to an equation, $2y^4-2=5y^2+10$. Trans. $2y^4-5y^2=12$. Completing the square by the second method, $16y^4-*+25=96+25=121$. Extracting the root, $4y^2-5=\sqrt{121}=11$. $4y^2=11+5=16$. $y^2=4$. $y=2$. Substituting the value of y in the 1st, then $4x=x+15$. $3x=15$. $x=5$. The series is, 5, 10, 20.

PROBLEM 7.

Let x, xy, xy^2, $xy^3=$ the series. By the conditions, 1st. $xy+24=xy^2$. 2d. $xy^3+x : xy^2+xy :: 7 : 3$. Div. the first couplet by x, then $y^3+1 : y^2+y :: 7 : 3$. Adding consequents to antecedents, $y^3+y^2+y+1 : y^2+y :: 10 : 3$. Div. the first couplet by $(y+1)$, then $y^2+1 : y :: 10 : 3$. Changing to an equation, $3y^2+3=10y$. Trans. $3y^2-10y=-3$. Completing the square by the second method, $36y^2-*+100=100-36=64$. Extracting the root, $6y-10=8$. $6y=18$. $y=3$. Substituting the value of y in the 1st, then $3x+24=27x$. $24=24x$. $1=x$. The series is, 1, 3, 9, 27.

COMPOUND DIVISION, AND GREATEST COMMON MEASURE.

☞ The author enters into no explanation of Compound Division, presuming that the student has followed through the process given in the Algebra. It is easily understood, and is similar to Simple Division, except that a *compound* instead of a *simple* divisor is employed.

Example 1.

$$\begin{array}{l} a+b\,)\,ac+bc+ad+bd\,(\,c+d. \\ \quad\;\; ac+bc \\ \quad\;\; \overline{*\quad *\quad ad+bd} \\ \qquad\qquad\;\; ad+bd \\ \qquad\qquad\;\; \overline{*\quad\;\; *} \end{array}$$

Example 2.

$$\begin{array}{l} a^2+ab+b^2\,)\,a^3+2a^2b+2ab^2+b^3\,(\,a+b. \\ \qquad\qquad\;\; a^3+\;a^2b+\;ab^2 \\ \qquad\qquad\;\; \overline{*\qquad a^2b+\;ab^2+b^3} \\ \qquad\qquad\qquad\;\; a^2b+\;ab^2+b^3 \\ \qquad\qquad\qquad\;\; \overline{*\qquad *\quad\;\; *} \end{array}$$

Example 3.

$$\begin{array}{l} 2a-y\,)\,6a^2x-3a^2xy-2a^2x+axy+2ax-xy\,(\,3a^2x-ax+x. \\ \qquad\quad 6a^2x-3a^2xy \\ \qquad\quad \overline{*\qquad *\quad -2a^2x+axy+2ax-xy} \\ \qquad\qquad\qquad\;\; -2a^2x+axy \\ \qquad\qquad\qquad\;\; \overline{\quad *\qquad *\quad\;\; 2ax-xy} \\ \qquad\qquad\qquad\qquad\qquad\quad 2ax-xy \\ \qquad\qquad\qquad\qquad\qquad\quad \overline{\;*\qquad *\;} \end{array}$$

☞ The star is used to denote cases in which the quantities immediately above them, in the minuend and subtrahend, are equal. Those quantities, cancelling each other, disappear from the operation.

EXAMPLE 4.

$$
\begin{array}{l}
a+x\,)\,a^3+x^3\,(\,a^2-ax+x^2. \\
\quad a^3+a^2x \\
\hline
\quad * \; -a^2x+x^3 \\
\quad\quad -a^2x-ax^2 \\
\hline
\quad\quad\quad * \; +ax^2+x^3 \\
\quad\quad\quad\quad +ax^2+x^3 \\
\hline
\quad\quad\quad\quad\quad * \quad *
\end{array}
$$

EXAMPLE 5.

$$
\begin{array}{l}
a-2ax+2x^2\,)\,a^4+4x^4\,(\,a^2+2ax+2x^2. \\
\quad a^4-2a^3x+2a^2x^2 \\
\hline
\quad * \; +2a^3x-2a^2x^2+4x^4 \\
\quad\quad +2a^3x-4a^2x^2+4ax^3 \\
\hline
\quad\quad\quad * \; +2a^2x^2-4ax^3+4x^4 \\
\quad\quad\quad\quad +2a^2x^2-4ax^3+4x^4 \\
\hline
\quad\quad\quad\quad\quad * \quad * \quad *
\end{array}
$$

EXAMPLE 6.

$$
\begin{array}{l}
a+1\,)\,a^3+a^2+a^2b+ab+3ac+3c\,(\,a^2+ab+3c. \\
\quad a^3+a^2 \\
\hline
\quad * \quad * \quad a^2b+ab+3ac+3c \\
\quad\quad\quad a^2b+ab \\
\hline
\quad\quad\quad * \quad * \quad 3ac+3c \\
\quad\quad\quad\quad\quad 3ac+3c \\
\hline
\quad\quad\quad\quad\quad * \quad *
\end{array}
$$

EXAMPLE 7.

$$
\begin{array}{l}
a+b-c)\,a+b-c-ax-bx+cx\,(\,1-x. \\
\quad a+b-c \\
\hline
\quad * \quad * \quad * -ax-bx+cx \\
\quad\quad\quad\quad -ax-bx+cx \\
\hline
\quad\quad\quad\quad\quad * \quad * \quad *
\end{array}
$$

EXAMPLE 8.

$$
\begin{array}{l}
2a^2-ax+x^2)\,2a^4-13a^3x+11a^2x^2-8ax^3+2x^4\,(a^2-6ax+2x^2.\\
\qquad\qquad 2a^4-\quad a^3x+\quad a^2x^2\\
\hline
\qquad\qquad * \;-12a^3x+10a^2x^2-8ax^3+2x^4\\
\qquad\qquad\quad -12a^3x+\;6a^2x^2-\;6ax^3\\
\hline
\qquad\qquad\qquad\quad * \;+\;4a^2x^2-\;2ax^3+2x^4\\
\qquad\qquad\qquad\qquad +\;4a^2x^2-\;2ax^3+2x^4\\
\hline
\qquad\qquad\qquad\qquad\quad * \qquad * \qquad *
\end{array}
$$

EXAMPLE 9.

$$
\begin{array}{l}
a+b)\,ac+bc+ad+bd+x\,(c+d+\dfrac{x}{a+b}.\\
\qquad ac+bc\\
\hline
\qquad * \quad * +ad+bd+x\\
\qquad\qquad\;\; +ad+bd\\
\hline
\qquad\qquad\quad * \quad * +x.
\end{array}
$$

EXAMPLE 10.

$$
\begin{array}{l}
d-h)\,ad-ah+bd-bh+y\,(a+b+\dfrac{y}{d-h}.\\
\qquad ad-ah\\
\hline
\qquad * \quad * +bd-bh+y\\
\qquad\qquad\;\; +bd-bh\\
\hline
\qquad\qquad\quad * \quad * +y.
\end{array}
$$

EXAMPLE 11.

$$
\begin{array}{l}
3a+y)\,6ax+2xy-3ab-by+3ac+cy+h\,(2x-b+c+\dfrac{h}{3a+y}.\\
\qquad 6ax+2xy\\
\hline
\qquad * \quad * \;-3ab-by+3ac+cy+h\\
\qquad\qquad\;\; -3ab-by\\
\hline
\qquad\qquad\quad * \quad * +3ac+cy+h\\
\qquad\qquad\qquad\quad +3ac+cy\\
\hline
\qquad\qquad\qquad\quad * \quad * +h.
\end{array}
$$

EXAMPLE 12.

$$b-3)\,a^2b-3a^2+2ab-6a-4b+22\,(a^2+2a-4+\frac{10}{b-3}.$$
$$\begin{array}{l}
a^2b-3a^2 \\
\hline
*\quad *\ +2ab-6a-4b+22 \\
\qquad\quad +2ab-6a \\
\hline
\qquad\qquad *\quad *\ -4b+22 \\
\qquad\qquad\qquad\ -4b+12 \\
\hline
\qquad\qquad\qquad\quad *\ +10.
\end{array}$$

EXAMPLE 13.

$$a+\sqrt{b})\,ac+c\sqrt{b}+a\sqrt{d}+\sqrt{bd}\,(c+\sqrt{d}.$$
$$\begin{array}{l}
ac+c\sqrt{b} \\
\hline
*\quad *\ +a\sqrt{d}+\sqrt{bd} \\
\qquad\quad +a\sqrt{d}+\sqrt{bd} \\
\hline
\qquad\qquad *\quad *
\end{array}$$

EXAMPLE 14.

$$a+\sqrt{y})\,a+\sqrt{y}+ar\sqrt{y}+ry\,(1+r\sqrt{y}.$$
$$\begin{array}{l}
a+\sqrt{y} \\
\hline
*\quad *\ +ar\sqrt{y}+ry \\
\qquad\quad +ar\sqrt{y}+ry \\
\hline
\qquad\qquad *\quad *
\end{array}$$

EXAMPLE 15.

$$x-a)\,x^3-3ax^2+3a^2x-a^3\,(x^2-2ax+a^2.$$
$$\begin{array}{l}
x^3-ax^2 \\
\hline
*\ -2ax^2+3a^2x-a^3 \\
\quad -2ax^2+2a^2x \\
\hline
\qquad *\ +\ a^2x-a^3 \\
\qquad\quad +\ a^2x-a^3 \\
\hline
\qquad\qquad *\quad *
\end{array}$$

EXAMPLE 16.

$$
\begin{array}{l}
y-8)2y^3-19y^2+26y-17\,(2y^2-3y+2-\dfrac{1}{y-8}.\\
\quad 2y^3-16y^2\\
\hline
\quad * \;-3y^2+26y-17\\
\qquad -3y^2+24y\\
\hline
\qquad\quad * \;+2y-17\\
\qquad\qquad +2y-16\\
\hline
\qquad\qquad\quad * \;-1.
\end{array}
$$

EXAMPLE 17.

$$
\begin{array}{l}
x-1)x^6-1\,(x^5+x^4+x^3+x^2+x+1.\\
\quad x^6-x^5\\
\hline
\quad * +x^5-1\\
\qquad +x^5-x^4\\
\hline
\qquad * +x^4-1\\
\qquad\quad +x^4-x^3\\
\hline
\qquad\quad * +x^3-1\\
\qquad\qquad +x^3-x^2\\
\hline
\qquad\qquad * +x^2-1\\
\qquad\qquad\quad +x^2-x\\
\hline
\qquad\qquad\quad * +x-1\\
\qquad\qquad\qquad +x-1\\
\hline
\qquad\qquad\qquad * \quad *
\end{array}
$$

EXAMPLE 18.

$$
\begin{array}{l}
2x^2+3x-1\,)\,4x^4-9x^2+6x-3\,(2x^2-3x+1-\dfrac{2}{2x^2+3x-1}.\\
\quad 4x^4+6x^3-2x^2\\
\hline
\quad * \;-6x^3-7x^2+6x-3\\
\qquad -6x^3-9x^2+3x\\
\hline
\qquad\quad * \;+2x^2+3x-3\\
\qquad\qquad +2x^2+3x-1\\
\hline
\qquad\qquad\quad * \quad * \;-2.
\end{array}
$$

EXAMPLE 19.

$$a+2b)a^4+4a^2b+3b^4(a^3-2a^2b+4ab+4ab^2-8b^2-8b^3+\frac{16b^3+19b^4}{a+2b}.$$

$$a^4+2a^3b$$

$$* -2a^3b+4a^2b+3b^4$$

$$-2a^3b-4a^2b^2$$

$$* +4a^2b+4a^2b^2+3b^4$$

$$+4a^2b+8ab^2$$

$$* +4a^2b^2-8ab^2+3b^4$$

$$+4a^2b^2+8ab^3$$

$$* -8ab^2-8ab^3+3b^4$$

$$-8ab^2-16b^3$$

$$* -8ab^3+16b^3+3b^4$$

$$-8ab^3-16b^4$$

$$* +16b^3+19b^4.$$

EXAMPLE 20.

$$x^2-ax+a^2)x^4-a^2x^2+2a^3x-a^4(x^2+ax-a^2.$$

$$x^4-ax^3+a^2x^2$$

$$* +ax^3-2a^2x^2+2a^3x-a^4$$

$$+ax^3-a^2x^2+a^3x$$

$$* -a^2x^2+a^3x-a^4$$

$$-a^2x^2+a^3x-a^4$$

$$* \quad * \quad *$$

GREATEST COMMON MEASURE.

EXAMPLE 1.

$$6a^2+7ax-3x^2)6a^2+11ax+3x^2(1.$$

$$6a^2+7ax-3x^2$$

$$* +4ax+6x^2.$$

Div. this remainder by $2x$ (Art. 466), we have as a new divisor,

$$\begin{array}{l} 2a+3x)\,6a^2+7ax-3x^2\,(3a-x. \\ \qquad\quad 6a^2+9ax \\ \qquad\quad\ \ * \ -2ax-3x^2 \\ \qquad\qquad\ -2ax-3x^2 \\ \qquad\qquad\quad * \qquad * \end{array}$$

$2a+3x=$ greatest common measure, as there is no remainder left after the last division.

Example 2.

$$\begin{array}{l} x^2+2bx+b^2)\,x^3-b^2x\,(x. \\ \qquad\qquad\qquad x^3+2bx^2+b^2x \\ \qquad\qquad\qquad\ * \ -2bx^2-2xb^2. \end{array}$$

Dividing this remainder by $-2bx$, we have as a new divisor,

$$\begin{array}{l} x+b)\,x^2+2bx+b^2\,(x+b. \\ \qquad x^2+\ bx \\ \qquad\ * + \ bx+b^2 \\ \qquad\quad + \ bx+b^2 \\ \qquad\qquad * \quad *. \end{array} \quad x+b=\text{G. C. M.}$$

Example 3.

Dividing (a^2c+a^2x) by a^2 according to Art. 466, we have as a divisor $(c+x)$; then

$$\begin{array}{l} c+x)\,cx+x^2\,(x. \\ \qquad\ cx+x^2 \\ \qquad\ \ * \quad *. \end{array} \quad c+x=\text{G. C. M.}$$

Example 4.

Dividing the first quantity by 3, and the second by 2, as follows:

$$\left.\begin{array}{l} 3)\,3x^3-24x-9 \\ \quad\ x^3-\ 8x-3 \end{array}\right\} \text{ and } \left\{\begin{array}{l} 2)\,2x^3-16x-6. \\ \quad\ x^3-\ 8x-3. \end{array}\right.$$

Both of these quantities being alike, then $x^3-8x-3=$ G. C. M.

This example can also be performed in the usual manner, by dividing the greater by the less.

EXAMPLE 5.

Dividing $(a^5-b^2a^3)$ by a^3, we have as a divisor (a^2-b^2). Dividing,

$$a^2-b^2)\,a^4-b^4\,(a^2+b^2.$$
$$\underline{a^4-a^2b^2}$$
$$\underline{*+a^2b^2-b^4}$$
$$\underline{+a^2b^2-b^4}$$
$$*\quad *.$$

Then $a^2-b^2=$ G. C. M.

EXAMPLE 6.

Dividing $(xy+y)$ by y, we have $(x+1)$ as a divisor; dividing by this,

$$x+1\,)\,x^2-1\,(x-1.$$
$$\underline{x^2+x}$$
$$\underline{*-x-1}$$
$$\underline{-x-1}$$
$$*\quad *.$$

Consequently $x+1=$ G. C. M.

EXAMPLE 7.

Dividing (x^4-a^4) by (x^2+a^2), the quotient (Art. 235) is (x^2-a^2). Dividing by this,

$$x^2-a^2)\,x^3-a^3\,(x.$$
$$\underline{x^3-a^2x}$$
$$*+a^2x-a^3.$$

Dividing this remainder by a^2, then the new divisor $=$

$$x-a)\,x^2-a^2\,(x+a.$$
$$\underline{x^2-ax}$$
$$\underline{*+ax-a^2}$$
$$\underline{+ax-a^2}$$
$$*\quad *.$$

$x-a=$ G. C. M.

EXAMPLE 8.

Dividing $(a^2-3ab+2b^2)$ by $(a-b)$, the quotient is $(a-2b)$. Taking this as the divisor,

$$
\begin{array}{l}
a-2b)a^2-ab-2b^2(a+b. \\
\quad a^2-2ab \\
\quad * +ab-2b^2 \\
\quad\quad +ab-2b^2 \\
\quad\quad * \quad *. \quad a-2b = \text{G. C. M.}
\end{array}
$$

EXAMPLE 9.

$$
\begin{array}{l}
a^3-a^2x-ax^2+x^3)a^4-x^4(a. \\
\quad\quad a^4-a^3x-a^2x^2+ax^3 \\
\quad\quad * +a^3x+a^2x^2-ax^3-x^4.
\end{array}
$$

Dividing this remainder by $(ax+x^2)$,

$$
\begin{array}{l}
ax+x^2)a^3x+a^2x^2-ax^3-x^4(a^2-x^2. \\
\quad a^3x+a^2x^2 \\
\quad * \quad * \quad -ax^3-x^4 \\
\quad\quad\quad -ax^3-x^4 \\
\quad\quad\quad * \quad *
\end{array}
$$

Taking (a^2-x^2) as the new divisor, and proceeding with the division,

$$
\begin{array}{l}
a^2-x^2)a^3-a^2x-ax^2+x^3(a-x. \\
\quad a^3 \quad\quad -ax^2 \\
\quad * -a^2x \quad * +x^3 \\
\quad\quad -a^2x \quad\quad +x^3 \\
\quad\quad * \quad\quad *. \quad a^2-x^2 = \text{G. C. M.}
\end{array}
$$

EXAMPLE 10.

Dividing (a^3-ab^2) by (a^2-ab),‡ we have $(a+b)$ as a quotient. Dividing the other by this,

‡
$$
\begin{array}{l}
a^2-ab)a^3-ab^2(a+b. \\
\quad a^3-a^2b \\
\quad * +a^2b-ab^2 \\
\quad\quad +a^2b-ab^2 \\
\quad\quad * \quad *
\end{array}
$$

$$
\begin{array}{l}
a+b)\,a^2+2ab+b^2\,(a+b. \\
\quad\;\; \underline{a^2+\;ab\qquad} \\
\quad\;\; *+\;ab+b^2 \\
\quad\;\;\; \underline{+\;ab+b\;\;} \\
\qquad\quad *\quad *. \quad a+b=\text{G. C. M.}
\end{array}
$$

☞ It should be remarked, that whenever either of the quantities, or any subsequent divisor or dividend, is divided by an assumed quantity, the provisions of Art. 466 are strictly followed.

BINOMIALS.

In the expansion of quantities under this head, the letters A, B, C, D, &c. will be used to denote the *undetermined* coefficients. The advantage of this will be apparent to the student after a few examples.

Example 1.

$(a^2+x)^{\frac{1}{2}}$. Let b represent a^2 in the operation, and we have

$$(b+x)^{\frac{1}{2}}=b^{\frac{1}{2}}+Ab^{-\frac{1}{2}}x+Bb^{-\frac{3}{2}}x^2+Cb^{-\frac{5}{2}}x^3+Db^{-\frac{7}{2}}x^4, \text{ \&c.}$$

$$A=\frac{1}{2}. \quad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

Restoring a^2 for b, using $\frac{1}{a}$ for a^{-1}, $\frac{1}{a^2}$ for a^{-2}, &c.; and substituting the coefficients for the letters, then the series will be as follows:

$$(a^2+x)^{\frac{1}{2}}=a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}, \text{ \&c.}$$

EXAMPLE 2.

$$(1+x)^{\frac{1}{2}}=1^{\frac{1}{2}}+A1^{-\frac{1}{2}}x+B1^{-\frac{3}{2}}x^2+C1^{-\frac{5}{2}}x^3+D1^{-\frac{7}{2}}x^4, \text{ \&c.}$$

$$A=\frac{1}{2}. \quad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

Disregarding the 1, as all its powers and roots are 1, the series will be,

$$(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}, \text{ \&c.}$$

EXAMPLE 3.

$$(1+1)^{\frac{1}{2}}=1^{\frac{1}{2}}+A1^{-\frac{1}{2}}+B1^{-\frac{3}{2}}+C1^{-\frac{5}{2}}+D1^{-\frac{7}{2}}+E1^{-\frac{9}{2}}, \text{ \&c.}$$

$$A=\frac{1}{2}. \quad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

$$E=-\frac{3.5}{2.4.6.8}\times\frac{-\frac{7}{2}}{5}=-\frac{3.5}{2.4.6.8}\times-\frac{7}{10}=\frac{3.5.7}{2.4.6.8.10}.$$

Series of $(1+1)^{\frac{1}{2}}=\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}$, &c.

EXAMPLE 4.

$$a^{\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=$$

$$a^{\frac{1}{2}}\times\left(1^{\frac{1}{2}}+A1^{-\frac{1}{2}}\frac{x}{a}+B1^{-\frac{3}{2}}\frac{x^2}{a^2}+C1^{-\frac{5}{2}}\frac{x^3}{a^3}+D1^{-\frac{7}{2}}\frac{x^4}{a^4}, \text{ \&c.}\right)$$

$$A=\frac{1}{2}.\quad B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

$$a^{\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{\frac{1}{2}}=a^{\frac{1}{2}}\times\left(1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4},\ \&c.\right)$$

Example 5.

$$a^{\frac{1}{3}}\times\left(1+\frac{b}{a}\right)^{\frac{1}{3}}=$$

$$a^{\frac{1}{3}}\left(1^{\frac{1}{3}}+A1^{-\frac{2}{3}}\frac{b}{a}+B1^{-\frac{5}{3}}\frac{b^2}{a^2}+C1^{-\frac{8}{3}}\frac{b^3}{a^3}+D1^{-\frac{11}{3}}\frac{b^4}{a^4},\ \&c.\right)$$

$$A=\frac{1}{3}.\quad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

$$D=\frac{2.5}{3.6.9}\times\frac{-\frac{8}{3}}{4}=\frac{2.5}{3.6.9}\times-\frac{8}{12}=-\frac{2.5.8}{3.6.9.12}.$$

$$a^{\frac{1}{3}}\times\left(1+\frac{b}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4},\ \&c.\right)$$

Example 6.

$$a^{\frac{1}{4}}\times\left(1-\frac{b}{a}\right)^{\frac{1}{4}}=$$

$$a^{\frac{1}{4}}\times\left(1^{\frac{1}{4}}-A1^{-\frac{3}{4}}\frac{b}{a}+B1^{-\frac{7}{4}}\frac{b^2}{a^2}-C1^{-\frac{11}{4}}\frac{b^3}{a^3}+D1^{-\frac{15}{4}}\frac{b^4}{a^4},\ \&c.\right)$$

$$A=\frac{1}{4}.\quad B=\frac{1}{4}\times\frac{-\frac{3}{4}}{2}=\frac{1}{4}\times-\frac{3}{8}=-\frac{3}{4.8}.$$

$$C=-\frac{3}{4.8}\times\frac{-\frac{7}{4}}{3}=-\frac{3}{4.8}\times-\frac{7}{12}=\frac{3.7}{4.8.12}.$$

$$D=\frac{3.7}{4.8.12}\times\frac{-\frac{11}{4}}{4}=\frac{3.7}{4.8.12}\times-\frac{11}{16}=-\frac{3.7.11}{4.8.12.16}.$$

As the terms which have a *plus* coefficient, also have a *minus* sign before them; consequently the signs of all except the first term will be *minus*.

$$a^{\frac{1}{4}}\times\left(1-\frac{b}{a}\right)^{\frac{1}{4}}=a^{\frac{1}{4}}\times\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12.a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}, \text{\&c.}\right)$$

EXAMPLE 7.

$(a+x)^{-\frac{1}{2}}=a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}$. Expanding, we have,

$$a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}=$$

$$a^{-\frac{1}{2}}\times\left(1^{-\frac{1}{2}}+A1^{-\frac{3}{2}}\frac{x}{a}+B1^{-\frac{5}{2}}\frac{x^2}{a^2}+C1^{-\frac{7}{2}}\frac{x^3}{a^3}+D1^{-\frac{9}{2}}\frac{x^4}{a^4}, \text{\&c.}\right)$$

$$A=-\frac{1}{2}.\quad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2.4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

$$D=-\frac{3.5}{2.4.6}\times\frac{-\frac{7}{2}}{4}=-\frac{3.5}{2.4.6}\times-\frac{7}{8}=\frac{3.5.7}{2.4.6.8}.$$

$$a^{-\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}=a^{-\frac{1}{2}}\times\left(1-\frac{x}{2a}+\frac{3x^2}{2.4a^2}-\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}, \text{\&c.}\right)$$

EXAMPLE 8.

$$(1-x)^{\frac{2}{5}}=1^{\frac{2}{5}}-A1^{-\frac{3}{5}}x+B1^{-\frac{8}{5}}x^2-C1^{-\frac{13}{5}}x^3+D1^{-\frac{18}{5}}x^4, \text{\&c.}$$

$$A=\frac{2}{5}.\quad B=\frac{2}{5}\times\frac{-\frac{3}{5}}{2}=\frac{2}{5}\times-\frac{3}{10}=-\frac{2.3}{5.10}.$$

$$C=-\frac{2.3}{5.10}\times\frac{-\frac{8}{5}}{3}=-\frac{2.3}{5.10}\times-\frac{8}{15}=\frac{2.3.8}{5.10.15}.$$

$$D=\frac{2.3.8}{5.10.15}\times\frac{-\frac{13}{5}}{4}=\frac{2.3.8}{5.10.15}\times-\frac{13}{20}=-\frac{2.3.8.13}{5.10.15.20}.$$

$$(1-x)^{\frac{2}{5}}=1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}, \text{ \&c.}$$

EXAMPLE 9.

$$(1+x)^{-\frac{1}{5}}=1^{-\frac{1}{5}}+A1^{-\frac{6}{5}}x+B1^{-\frac{11}{5}}x^2+C1^{-\frac{17}{5}}x^3, \text{ \&c.}$$

$$A=-\frac{1}{5}.\quad B=-\frac{1}{5}\times\frac{-\frac{6}{5}}{2}=-\frac{1}{5}\times-\frac{6}{10}=\frac{6}{5.10}.$$

$$C=\frac{6}{5.10}\times\frac{-\frac{11}{5}}{3}=\frac{6}{5.10}\times-\frac{11}{15}=-\frac{6.11}{5.10.15}.$$

$$(1+x)^{-\frac{1}{5}}=1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}, \text{ \&c.}$$

EXAMPLE 10.

$(a^2+x)^{-\frac{1}{2}}$. Substitute b for a^2; expanding, we have,

$$(b+x)^{-\frac{1}{2}}=b^{-\frac{1}{2}}+Ab^{-\frac{3}{2}}x+Bb^{-\frac{5}{2}}x^2+Cb^{-\frac{7}{2}}x^3+Db^{-\frac{9}{2}}x^4, \text{ \&c.}$$

$$A=-\frac{1}{2}.\quad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2.4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

$$D=-\frac{3.5}{2.4.6}\times\frac{-\frac{7}{2}}{4}=-\frac{3.5}{2.4.6}\times-\frac{7}{8}=\frac{3.5.7}{2.4.6.8}.$$

Restoring the value of b, and using coefficients, we have,

$$(a^2+x)^{-\frac{1}{2}}=\frac{1}{a}-\frac{x}{2a^3}+\frac{3x^2}{2.4a^5}-\frac{3.5x^3}{2.4.6a^7}+\frac{3.5.7x^4}{2.4.6.8a^9}, \text{ \&c.}$$

☞ It should be remembered, that quantities with *negative* indices are most properly and generally expressed as fractions; for instance, $a^{-3}=\frac{1}{a^3}$, or $2b^{-4}=\frac{2}{b^4}$. For further examples, see Art. 207.

PROMISCUOUS EXAMPLES.

EXAMPLE 1.

Required the 8th power of $(a+b)$. The terms are as follows:

Without coefficients,	$a^8,\ a^7b,\ a^6b^2,\ a^5b^3,\ a^4b^4,\ a^3b^5,\ a^2b^6,\ ab^7,\ b^8.$
The coefficients are,	1, 8, 28, 56, 70, 56, 28, 8, 1.

The terms complete are,

$$a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8.$$

EXAMPLE 2.

Required the 7th power of $(a-b)$. The terms are as follows:

Without coefficients,	$a^7,\ a^6b,\ a^5b^2,\ a^4b^3,\ a^3b^4,\ a^2b^5,\ ab^6,\ b^7.$
The coefficients are,	1, 7, 21, 85, 35, 21, 7, 1.

The terms complete are,

$$a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7.$$

EXAMPLE 3.

$$(1-a)^{-1}=1^{-1}-A1^{-2}a+B1^{-3}a^2-C1^{-4}a^3+D1^{-5}a^4-E1^{-6}a^5, \text{ \&c.}$$

$$A=-1. \quad B=-1\times\frac{-2}{2}=1. \quad C=1\times\frac{-3}{3}=-1.$$

$$D=-1\times\frac{-4}{4}=1. \quad E=1\times\frac{-5}{5}=-1.$$

As the minus coefficients change the minus terms to plus, the series will be,

$$(1-a)^{-1}=1+a+a^2+a^3+a^4+a^5, \text{ \&c.}$$

EXAMPLE 4.

$$h\times(a-b)^{-1}=h\times(a^{-1}-Aa^{-2}b+Ba^{-3}b^2-Ca^{-4}b^3+Da^{-5}b^4, \text{ \&c.})$$

$$A=-1. \quad B=-1\times\frac{-2}{2}=+1.$$

$$C=1\times\frac{-3}{3}=-1. \quad D=-1\times\frac{-4}{4}=+1.$$

$$h(a-b)^{-1}=h\times\left(\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}, \text{\&c.}\right)$$

Example 5.

Let c represent a^2; and d, b^2:

$$(c+d)^{\frac{1}{2}}=c^{\frac{1}{2}}+Ac^{-\frac{1}{2}}d+Bc^{-\frac{3}{2}}d^2+Cc^{-\frac{5}{2}}d^3+Dc^{-\frac{7}{2}}d^4, \text{\&c.}$$

Of course the coefficients are the same as in Exs. 1, 2, 3, 4 (Art. 482), and

$$A=\frac{1}{2}. \quad B=-\frac{1}{2.4}=-\frac{1}{8}.$$

$$C=\frac{3}{2.4.6}=\frac{1}{16}. \quad D=-\frac{3.5}{2.4.6.8}=-\frac{5}{128}.$$

Restoring the values of c and d, viz. a^2 and b^2, then

$$(a^2+b^2)^{\frac{1}{2}}=a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}-\frac{5b^8}{128a^7}, \text{\&c.}$$

Example 6.

$$(a+y)^{-4}=a^{-4}+Aa^{-5}y+Ba^{-6}y^2+Ca^{-7}y^3+Da^{-8}y^4, \text{\&c.}$$

$$A=-4. \quad B=-4\times\frac{-5}{2}=10.$$

$$C=10\times\frac{-6}{3}=-20. \quad D=-20\times\frac{-7}{4}=35.$$

$$(a+y)^{-4}=\frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}, \text{\&c.}$$

Example 7.

Substituting a for c^3 and b for x^3, then

$$(a+b)^{\frac{1}{3}}=a^{\frac{1}{3}}+Aa^{-\frac{2}{3}}b+Ba^{-\frac{5}{3}}b^2+Ca^{-\frac{8}{3}}b^3, \text{\&c.}$$

$$A=\frac{1}{3}. \quad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

Restoring the values of a and b, viz. c^3 and x^3, then

$$(c^3+x^3)^{\frac{1}{3}}=c+\frac{x^3}{3c^2}-\frac{2x^6}{3.6c^5}+\frac{2.5x^9}{3.6.9c^8}, \text{ \&c.}$$

Or bringing c out of the series, and placing it before the vinculum,

$$(c^3+x^3)^{\frac{1}{3}}=c\times\left(1+\frac{x^3}{3c^3}-\frac{2x^6}{3.6c^6}+\frac{2.5x^9}{3.6.9c^9}, \text{ \&c.}\right)$$

EXAMPLE 8.

$d(c^2+x^2)^{-\frac{1}{2}}$. Substitute a for c^2, and b for x^2, then

$$d(a+b)^{-\frac{1}{2}}=d\times(a^{-\frac{1}{2}}+Aa^{-\frac{3}{2}}b+Ba^{-\frac{5}{2}}b^2+Ca^{-\frac{7}{2}}b^3, \text{ \&c.})$$

$$A=-\frac{1}{2}.\quad B=-\frac{1}{2}\times\frac{-\frac{3}{2}}{2}=-\frac{1}{2}\times-\frac{3}{4}=\frac{3}{2.4}.$$

$$C=\frac{3}{2.4}\times\frac{-\frac{5}{2}}{3}=\frac{3}{2\cdot4}\times-\frac{5}{6}=-\frac{3.5}{2.4.6}.$$

Restoring the values of a and b, viz. c^2+x^2, we have,

$$d(c^2+x^2)^{-\frac{1}{2}}=d\times\left(\frac{1}{c}-\frac{x^2}{2c^3}+\frac{3x^4}{2.4c^5}-\frac{3.5x^6}{2.4.6c^7}, \text{ \&c.}\right)$$

Or bringing c out of the denominators of the series, then

$$d(c^2+x^2)^{-\frac{1}{2}}=\frac{d}{c}\times\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2.4c^4}-\frac{3.5x^6}{2.4.6c^6}, \text{ \&c.}\right)$$

EXAMPLE 9.

$(a^2+y^3)^5$. Let c represent a^2, and x represent y^3:

$$(c+x)^5=c^5+5c^4x+10c^3x^2+10c^2x^3+5cx^4+x^5.$$

Restoring the values of c and x, viz. a^2 and y^3, then

$$(a^2+y^3)^5=a^{10}+5a^8y^3+10a^6y^6+10a^4y^9+5a^2y^{12}+y^{15}.$$

EXAMPLE 10.

$(a+b+x)^4$. Let c represent $(b+x)$.

$$(a+c)^4=a^4+4a^3c+6a^2c^2+4ac^3+c^4. \quad \text{Restoring } (b+x),$$

$$(a+b+x)^4=a^4+4a^3(b+x)+6a^2(b+x)^2+4a(b+x)^3+(b+x)^4.$$

Expanding and involving the terms which have $(b+x)$,

$$(a+b+x)^4 = a^4+4a^3b+4a^3x+6a^2b^2+12a^2bx+6a^2x^2+4ab^3+12ab^2x+12abx^2+4ax^3+b^4+4b^3x+6b^2x^2+4bx^3+x^4.$$

EXAMPLE 11.

$(a^3-x)^{\frac{1}{2}}$. Substitute b for a^3, and we have

$$(b-x)^{\frac{1}{2}}=b^{\frac{1}{2}}-Ab^{-\frac{1}{2}}x+Bb^{-\frac{3}{2}}x^2-Cb^{-\frac{5}{2}}x^3+Db^{-\frac{7}{2}}x^4, \text{ \&c.}$$

The coefficients are the same as in Exs. 1, 2, &c. (Art. 482), for the indices are similar.

$$A=\frac{1}{2}. \quad B=-\frac{1}{2.4}. \quad C=\frac{3}{2.4.6}. \quad D=-\frac{3.5}{2.4.6.8}.$$

Restoring the value of b, then the series will be,

$$(a^3-x)^{\frac{1}{2}}=a^{\frac{3}{2}}-\frac{x}{2a^{\frac{3}{2}}}-\frac{x^2}{2.4a^{\frac{9}{2}}}-\frac{3x^3}{2.4.6a^{\frac{15}{2}}}-\frac{3.5x^4}{2.4.6.8a^{\frac{21}{2}}}, \text{ \&c.}$$

EXAMPLE 12.

$$(1-y^2)^{\frac{1}{2}}=1^{\frac{1}{2}}-A1^{-\frac{1}{2}}y^2+B1^{-\frac{3}{2}}y^4-C1^{-\frac{5}{2}}y^6+D1^{-\frac{7}{2}}y^8, \text{ \&c.}$$

$$A=\frac{1}{2}. \quad B=-\frac{1}{2.4}. \quad C=\frac{3}{2.4.6}. \quad D=-\frac{3.5}{2.4.6.8}.$$

$$(1-y^2)^{\frac{1}{2}}=1-\frac{y^2}{2}-\frac{y^4}{2.4}-\frac{3y^6}{2.4.6}-\frac{3.5y^8}{2.4.6.8}, \text{ \&c.}$$

The value of the coefficients is obtained by substituting the first power of some letter, as a for y^2; and thus the indices being similar to those of the preceding example, the coefficients are the same.

EXAMPLE 13.

$$(a-x)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}.$$

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1^{\frac{1}{3}}-A1^{-\frac{2}{3}}\frac{x}{a}+B1^{-\frac{5}{3}}\frac{x^2}{a^2}-C1^{-\frac{8}{3}}\frac{x^3}{a^3},\text{ \&c.}\right)$$

$$A=\frac{1}{3}.\quad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

$$a^{\frac{1}{3}}\times\left(1-\frac{x}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\times\left(1-\frac{x}{3a}-\frac{2x^2}{3.6a^2}-\frac{2.5x^3}{3.6.9a^3},\text{ \&c.}\right)$$

EXAMPLE 14.

$h(a^3-y^3)^{\frac{1}{3}}$. Let x stand for a^3, and b for y^3.

$$h(x-b)^{\frac{1}{3}}=h\times(x^{\frac{1}{3}}-Ax^{-\frac{2}{3}}b+Bx^{-\frac{5}{3}}b^2-Cx^{-\frac{8}{3}}b^3,\text{ \&c.})$$

$$A=\frac{1}{3}.\quad B=\frac{1}{3}\times\frac{-\frac{2}{3}}{2}=\frac{1}{3}\times-\frac{2}{6}=-\frac{2}{3.6}.$$

$$C=-\frac{2}{3.6}\times\frac{-\frac{5}{3}}{3}=-\frac{2}{3.6}\times-\frac{5}{9}=\frac{2.5}{3.6.9}.$$

Restoring the values of x and b, we have,

$$h(a^3-y^3)^{\frac{1}{3}}=h\times\left(a-\frac{y^3}{3a^2}-\frac{2y^6}{3.6a^5}-\frac{2.5y^9}{3.6.9a^8},\text{ \&c.}\right)$$

EVOLUTION OF COMPOUND QUANTITIES.

Example 1.

Required the cube root of $(a^6+3a^5-3a^4-11a^3+6a^2+12a-8)$.

$$
\begin{array}{rl}
 & a^6+3a^5-3a^4-11a^3+6a^2+12a-8\,(a^2+a-2 \\
 & a^6 \\
\hline
3a^4) & * \quad 3a^5-3a^4-11a^3+6a^2+12a-8 \\
\hline\hline
 & a^6+3a^5-3a^4-11a^3+6a^2+12a-8 \\
 & a^6+3a^5+3a^4+\;\; a^3= \qquad\qquad (a^2+a)^3. \\
\hline
3a^4) & * \quad * \; -6a^4-12a^3+6a^2+12a-8 \\
\hline\hline
 & a^6+3a^5-3a^4-11a^3+6a^2+12a-8 \\
 & a^6+3a^5-3a^4-11a^3+6a^2+12a-8=(a^2+a-2)^3. \\
\hline
 & * \quad * \quad * \quad * \quad * \quad * \quad *
\end{array}
$$

Example 2.

Required the fourth root of $(a^4+8a^3+24a^2+32a+16)$.

$$
\begin{array}{rl}
 & a^4+8a^3+24a^2+32a+16\,(a+2 \\
 & a^4 \\
\hline
4a^3) & * \quad 8a^3+24a^2+32a+16 \\
\hline\hline
 & a^4+8a^3+24a^2+32a+16 \\
 & a^4+8a^3+24a^2+32a+16=(a+2)^4. \\
\hline
 & * \quad * \quad * \quad * \quad *
\end{array}
$$

Example 3.

The fifth root of $(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5)$.

$$
\begin{array}{rl}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\,(a+b \\
 & a^5 \\
\hline
5a^4) & * \quad 5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
\hline\hline
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=(a+b)^5. \\
\hline
 & * \quad * \quad * \quad * \quad * \quad *
\end{array}
$$

EXAMPLE 4.

The cube root of $(a^3-6a^2b+12ab^2-8b^3)$.

$$
\begin{array}{rl}
 & a^3-6a^2b+12ab^2-8b^3\,(a-2b \\
 & a^3 \\
3a^2) & *-6a^2b+12ab^2-8b^3 \\
 & a^3-6a^2b+12ab^2-8b^3 \\
 & a^3-6a^2b+12ab^2-8b^3=(a-2b)^3. \\
 & *\quad *\quad *\quad *
\end{array}
$$

EXAMPLE 5.

The square root of $(4a^2-12ab+9b^2+16ah-24bh+16h^2)$.

$$
\begin{array}{rl}
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2\,(2a-3b+4h. \\
 & 4a^2 \\
4a) & *\ -12ab+9b^2+16ah-24bh+16h^2 \\
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2 \\
 & 4a^2-12ab+9b^2= \qquad (2a-3b)^2. \\
4a) & *\ -\ *\quad *\quad 16ah-24bh+16h^2 \\
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2 \\
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2=(2a-3b+4h)^2. \\
 & *\quad *\quad *\quad *\quad *\quad *
\end{array}
$$

EXAMPLE 3. (UNDER SQUARE ROOT).

$$
\begin{array}{rl}
 & a^6-2a^5+3a^4-2a^3+a^2\,(a^3-a^2+a. \\
 & a^6 \\
2a^3-a^2) & *-2a^5+3a^4-2a^3+a^2 \\
 & -2a^5+a^4 \\
2a^3-2a^2+a) & *\ +2a^4-2a^3+a^2 \\
 & +2a^4-2a^3+a^2 \\
 & *\quad *\quad *
\end{array}
$$

Example 4.

$$a^4+4a^2b+4b^2-4a^2-8b+4\,(a^2+2b-2.$$
$$a^4$$
$$2a^2+2b)\;*\;+4a^2b+4b^2-4a^2-8b+4$$
$$4a^2b+4b^2$$
$$2a^2+4b-2)\;*\quad *\;-4a^2-8b+4$$
$$-4a^2-8b+4.$$
$$*\quad *\quad *$$

PROMISCUOUS EXAMPLES.

Example 1.

Required the square root of $(x^4-4x^3+6x^2-4x+1)$.

$$x^4-4x^3+6x^2-4x+1\,(x^2-2x+1.$$
$$x^4$$
$$2x^2-2x)\;*\;-4x^3+6x^2-4x+1$$
$$-4x^3+4x^2$$
$$2x^2-4x+1)\quad *\quad 2x^2-4x+1$$
$$2x^2-4x+1$$
$$*\quad *\quad *$$

Example 2.

Required the cube root of $(x^6-6x^5+15x^4-20x^3+15x^2-6x+1)$.

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,(x^2-2x+1.$$
$$x^6$$
$$3x^4)\;*\;-6x^5+15x^4-20x^3+15x^2-6x+1$$
$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1$$
$$x^6-6x^5+12x^4-\;8x^3= \qquad (x^2-2x)^3.$$
$$3x^4)\;*\quad *\quad 3x^4-12x^3+15x^2-6x+1$$
$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1$$
$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1=(x^2-2x+1)^3.$$
$$*\quad *\quad *\quad *\quad *\quad *\quad *$$

EXAMPLE 3.

Required the square root of $(4x^4-4x^3+13x^2-6x+9)$.

$$4x^4-4x^3+13x^2-6x+9\,(2x^2-x+3.$$
$$4x^4$$
$$4x^2-x)\ *\ -4x^3+13x^2-6x+9$$
$$-4x^3+\ x^2$$
$$4x^2-2x+3)\quad *\ +12x^2-6x+9$$
$$+12x^2-6x+9$$
$$*\quad *\quad *$$

EXAMPLE 4.

The 4th root of $(16a^4-96a^3x+216a^2x^2-216ax^3+81x^4)$.

$$16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\,(2a-3x.$$
$$16a^4$$
$$32a^3)\ *\ -96a^3x+216a^2x^2-216ax^3+81x^4$$
$$16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$$
$$16a^4-96a^3x+216a^2x^2-216ax^3+81x^4=(2a-3x)^4.$$
$$*\quad *\quad *\quad *\quad *$$

EXAMPLE 5.

The 5th root of $(x^5+5x^4+10x^3+10x^2+5x+1)$.

$$x^5+5x^4+10x^3+10x^2+5x+1\,(x+1.$$
$$x^5$$
$$5x^4)\ *\ 5x^4+10x^3+10x^2+5x+1$$
$$x^5+5x^4+10x^3+10x^2+5x+1$$
$$x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5.$$
$$*\quad *\quad *\quad *\quad *\quad *$$

EXAMPLE 6.

The 6th root of $(a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6)$.

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\,(a-b.$$
$$a^6$$
$$6a^5)\;*\;-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$
$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$
$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6=(a-b)^6.$$
$$*\quad *\quad *\quad *\quad *\quad *$$

ROOTS OF BINOMIAL SURDS.

EXAMPLE 1.

The square root of $3+2\sqrt{2}$. $a=3$. $\sqrt{b}=2\sqrt{2}$. Then $a^2=9$. $b=(2\sqrt{2})^2=8$. $a^2-b=9-8=1$. By formula 1st, $\sqrt{3+2\sqrt{2}}=\sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}}$. $\sqrt{\frac{3+1}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}$. $\sqrt{\frac{3-1}{2}}=\sqrt{\frac{2}{2}}=\sqrt{1}=1$. Then $\sqrt{3+2\sqrt{2}}=\sqrt{2}+1$.

EXAMPLE 2.

The square root of $11+6\sqrt{2}$. $a=11$. $\sqrt{b}=6\sqrt{2}$. Then $a^2=121$. $b=(6\sqrt{2})^2=72$. $a^2-b=121-72=49$. $\sqrt{a^2-b}=\sqrt{49}=7$. By formula 1st, we have, $\sqrt{11+6\sqrt{2}}=\sqrt{\frac{11+7}{2}}+\sqrt{\frac{11-7}{2}}$. $\sqrt{\frac{11+7}{2}}=\sqrt{\frac{18}{2}}=\sqrt{9}=3$. $\sqrt{\frac{11-7}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}$. Then $\sqrt{11+6\sqrt{2}}=3+\sqrt{2}$.

EXAMPLE 3.

The square root of $6-2\sqrt{5}$. $a=6$. $\sqrt{b}=2\sqrt{5}$. $a^2=36$. $b=(2\sqrt{5})^2=20$. $a^2-b=36-20=16$. $\sqrt{a^2-b}=\sqrt{16}=4$. By formula 2d, $\sqrt{6-2\sqrt{5}}=\sqrt{\frac{6+4}{2}}-\sqrt{\frac{6-4}{2}}$.

$\sqrt{\frac{6+4}{2}}=\sqrt{\frac{10}{2}}=\sqrt{5}$. $\sqrt{\frac{6-4}{2}}=\sqrt{\frac{2}{2}}=\sqrt{1}=1$. Then $\sqrt{6-2\sqrt{5}}=\sqrt{5}-1$.

Example 4.

The square root of $7+4\sqrt{3}$. $a=7$. $\sqrt{b}=4\sqrt{3}$. $a^2=49$. $b=(4\sqrt{3})^2=48$. $a^2-b=49-48=1$. By formula 1st, $\sqrt{7+4\sqrt{3}}=\sqrt{\frac{7+1}{2}}+\sqrt{\frac{7-1}{2}}$. $\sqrt{\frac{7+1}{2}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2$. $\sqrt{\frac{7-1}{2}}=\sqrt{\frac{6}{2}}=\sqrt{3}$. Then $\sqrt{7+\sqrt{3}}=2+\sqrt{3}$.

Example 5.

The square root of $7-2\sqrt{10}$. $a=7$. $\sqrt{b}=2\sqrt{10}$. $a^2=49$. $b=(2\sqrt{10})^2=40$. $a^2-b=49-40=9$. $\sqrt{a^2-b}=\sqrt{9}=3$. By formula 2d, $\sqrt{7-2\sqrt{10}}=\sqrt{\frac{7+3}{2}}-\sqrt{\frac{7-3}{2}}$. $\sqrt{\frac{7+3}{2}}=\sqrt{\frac{10}{2}}=\sqrt{5}$. $\sqrt{\frac{7-3}{2}}=\sqrt{\frac{4}{2}}=\sqrt{2}$. Then $\sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}$.

INFINITE SERIES.

Example 1.

Reduce $\sqrt{a^2+b^2}$ to an infinite series. By Art. 485,

$$a^2+b^2\left(a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}, \text{ \&c.}\right.$$

$$a^2$$

$$\left.2a+\frac{b^2}{2a}\right)\ * \ +b^2$$

$$+b^2+\frac{b^4}{4a^2}$$

$$\left.2a+\frac{b^2}{a}-\frac{b^4}{8a^3}\right)\ * \ -\frac{b^4}{4a^2}$$

$$-\frac{b^4}{4a^2}-\frac{b^6}{8a^4}+\frac{b^8}{64a^6}$$

$$\left.2a+\frac{b^2}{a}-\frac{b^4}{4a^3}+\frac{b^6}{16a^5}\right)\ * \ +\frac{b^6}{8a^4}-\frac{b^8}{64a^6}, \text{ \&c. \&c.}$$

Example 2.

Reduce $\sqrt{a^2-b^2}$ to an infinite series.

$$a^2-b^2\left(a-\frac{b^2}{2a}-\frac{b^4}{8a^3}-\frac{b^6}{16a^5}, \text{ \&c.}\right.$$

$$a^2$$

$$\left.2a-\frac{b^2}{2a}\right)\ * \ -b^2$$

$$-b^2+\frac{b^4}{4a^2}$$

$$\left.2a-\frac{b^2}{a}-\frac{b^4}{8a^3}\right)\ * \ -\frac{b^4}{4a^2}$$

$$-\frac{b^4}{4a^2}+\frac{b^6}{8a^4}+\frac{b^8}{64a^6}$$

$$\left.2a-\frac{b^2}{a}-\frac{b^4}{4a^3}-\frac{b^6}{16a^5}\right)\ * \ -\frac{b^6}{8a^4}-\frac{b^8}{64a^6}, \text{ \&c.}$$

Example 3.

Reduce $\sqrt{2}$, or $\sqrt{1+1}$, to an infinite series.

$$1+1\left(1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}, \text{ \&c.}\right.$$

$$\begin{array}{rl}
 & \underline{1} \\
2+\frac{1}{2}) & *+1 \\
 & \underline{+1+\frac{1}{4}} \\
2+1-\frac{1}{8}) & *-\frac{1}{4} \\
 & \underline{-\frac{1}{4}-\frac{1}{8}+\frac{1}{64}} \\
2+1-\frac{1}{4}+\frac{1}{16}) & *+\frac{1}{8}-\frac{1}{64}, \text{ \&c. \&c.}
\end{array}$$

Example 4.

Reduce $\sqrt{1+x}$ to an infinite series.

$$1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}, \text{ \&c.}\right.$$

$$\begin{array}{rl}
 & \underline{1} \\
2+\frac{x}{2}) & *+x \\
 & \underline{+x+\frac{x^2}{4}} \\
2+x-\frac{x^2}{8}) & *-\frac{x^2}{4} \\
 & \underline{-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}} \\
2+x-\frac{x^2}{8}+\frac{x^3}{16}) & *+\frac{x^3}{8}-\frac{x^4}{64}, \text{ \&c. \&c.}
\end{array}$$

INDETERMINATE COEFFICIENTS.

EXAMPLE 3.

$\frac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4$, &c. Clearing of fractions, $1+2x=A-Ax-Ax^2+Bx-Bx^2-Bx^3+Cx^2-Cx^3-Cx^4+Dx^3-Dx^4-Dx^5+Ex^4-Ex^5$, &c. Trans. $(1+2x)$, we have $0=(A-1)+(B-A-2)x+(C-A-B)x^2+(D-C-B)x^3+(E-D-C)x^4$, &c. $A-1=0$. $A=1$. $B-A-2=0$. $B=A+2=1+2=3$. $C-A-B=0$. $C=A+B=1+3=4$. $D-C-B=0$. $D=C+B=4+3=7$. $E-D-C=0$. $E=D+C=7+4=11$. Then, by substitution, $\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4$, &c. It will be seen that the indices increase regularly by the addition of 1; and that the coefficient of any term is equal to the sum of the coefficients of the two preceding terms. Then the following terms $=18x^5+29x^6+47x^7$, &c.

EXAMPLE 4.

$\frac{d}{b-ax}=A+Bx+Cx^2+Dx^3$, &c. Clearing of fractions, $d=Ab-Aax+Bbx-Bax^2+Cbx^2-Cax^3+Dbx^3-Dax^4$, &c. Transposing d, then $0=(Ab-d)+(Bb-Aa)x+(Cb-Ba)x^2+(Db-Ca)x^3$, &c. $Ab-d=0$. $Ab=d$. $A=\frac{d}{b}$. $Bb-Aa=0$. $Bb=Aa=\frac{ad}{b}$. $B=\frac{ad}{b^2}$. $Cb-Ba=0$. $Cb=Ba=\frac{a^2d}{b^2}$. $C=\frac{a^2d}{b^3}$. It will be observed that each coefficient multiplied by $\left(\frac{a}{b}\right)$ gives the succeeding one. Consequently the series are,

$$\frac{d}{b}+\frac{adx}{b^2}+\frac{a^2dx^2}{b^3}+\frac{a^3dx^3}{b^4}+\frac{a^4dx^4}{b^5}+\frac{a^5dx^5}{b^6}, \text{ \&c.}$$

Bringing the factor $\left(\frac{d}{b}\right)$ out of the series, it will remain as follows:

$$\frac{d}{b-ax}=\frac{d}{b}\left(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}=\frac{a^5x^5}{b^5}, \text{\&c.}\right)$$

Example 5.

$\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2+Dx^3$, &c. Clearing of fractions, $1-x=A-2Ax-3Ax^2+Bx-2Bx^2-3Bx^3+Cx^2-2Cx^3-3Cx^4+Dx^3$, &c. Transposing $(1-x)$, then $0=(A-1)+(1+B-2A)x+(C-2B-3A)x^2+(D-2C-3B)x^3$. $A-1=0$. $A=1$. $B+1-2A=0$. $B=+2A-1=2-1=1$. $C-2B-3A=0$. $C=2B+3A=2+3=5$. $D-2C-3B=0$. $D=2C+3B=10+3=13$. The law which governs the relation of coefficients, though more abstruse than in preceding cases, is equally certain. It is: The coefficient of any term is equal to twice the coefficient of the preceding term added to three times the coefficient of the term immediately preceding that; that is, 3d=2(2d)+3(1st). Then $E=2D+3C=2\times13+3\times5=41$. $F=2E+3D=2\times41+3\times13=121$. $G=2F+3E=2\times121+3\times41=365$.

$$\frac{1-x}{1-2x-3x^2}=1+x+5x^2+13x^3+41x^4+121x^5+365x^6, \text{\&c.}$$

Example 6.

$\frac{1}{1-x-x^2+x^3}=A+Bx+Cx^2+Dx^3+Ex^4$, &c. Clearing of fractions, $1=A-Ax-Ax^2+Ax^3+Bx-Bx^2-Bx^3+Bx^4+Cx^2-Cx^3-Cx^4+Cx^5+Dx^3-Dx^4-Dx^5+Dx^6+Ex^4$, &c. Transposing (1), then $0=(A-1)+(B-A)x+(C-A-B)x^2+(D+A-B-C)x^3+(E+B-C-D)x^4$, &c. $A-1=0$. $A=1$. $B-A=0$. $B=A=1$. $C-A-B=0$. $C=A+B=2$. $D+A-B-C=0$. $D=B+C-A=1+2-1=2$. $E+B-C-D=0$. $E=C+D-B=2+2-1=3$. The coefficients here increase by the addition of one in the alternate terms; or the law more correctly expressed is: The coefficient of any term is equal to the sum of the coefficients of the two preceding terms, minus the coefficient of the third preceding term; viz. 3d+2d−1st=4th. Then $F=E+D-C=3+2-2=3$. $G=F+E-D=3+3-2=4$.

$$\frac{1}{1-x-x^2+x^3}=1+x+2x^2+2x^3+3x^4+3x^5+4x^6, \text{ \&c.}$$

EXAMPLE 7.

$\frac{a}{1-bx}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$, &c. Clearing of fractions, $a=A-Abx+Bx-Bbx^2+Cx^2-Cbx^3+Dx^3-Dbx^4+Ex^4-Ebx+Fx^5$, &c. Transposing a, we have $0=(A-a)+(B-Ab)x+(C-Bb)x^2+(D-Cb)x^3+(E-Db)x^4$, &c. $A-a=0$. $A=a$. $B-Ab=0$. $B=Ab=ab$. $C-Bb=0$. $C=Bb=ab^2$. $D-Cb=0$. $D=Cb=ab^3$. Here the coefficients increase by the multiple b, and the series $=a+abx+ab^2x^2+ab^3x^3+ab^4x^4+ab^5x^5$, &c. Bringing out the a from the series, we have

$$\frac{a}{1-bx}=a\times(1+bx+b^2x^2+b^3x^3+b^4x^4+b^5x^5, \text{ \&c.})$$

EXAMPLE 8.

$\frac{1-x}{1-5x+6x^2}=A+Bx+Cx^2+Dx^3+Ex^4$, &c. Clearing of fractions, $1-x=A-5Ax+6Ax^2+Bx-5Bx^2+6Bx^3+Cx^2-5Cx^3+6Cx^4+Dx^3-5Dx^4+6Dx^5+Ex^4$, &c. Trans. $(1-x)$, then $0=(A-1)+(B+1-5A)x+(C+6A-5B)x^2+(D+6B-5C)x^3+(E+6C-5D)x^4$, &c. $A-1=0$. $A=1$. $B+1-5A=0$. $B=5A-1=5-1=4$. $C+6A-5B=0$. $C=5B-6A=20-6=14$. The law of progression in the coefficients may be briefly stated as follows: 3d=5(2d)−6(1st). Then $D=5\times14-6\times4=70-24=46$. $E=5\times46-6\times14=230-84=146$.

$$\frac{1-x}{1-5x+6x^2}=1+4x+14x^2+46x^3+146x^4, \text{ \&c.}$$

EXAMPLE 9.

$$\frac{a+bx}{(1-dx)^2}=\frac{a+bx}{1-2dx+d^2x^2}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5, \text{ \&c.}$$

Clearing of fractions, $a+bx=A-2Adx+Ad^2x^2+Bx-2Bdx^2+Bd^2x^3+Cx^2-2Cdx^3+Cd^2x^4+Dx^3-2Ddx^4+Dd^2x^5+Ex^4$, &c. Transposing $(a+bx)$, then $0=(A-a)+(B-b-2Ad)x+(C+Ad^2-2Bd)x^2+(D+Bd^2-2Cd)x^3+(E+Cd^2-2Dd)x^4$, &c. $A-a=0$. $A=a$. $B-b-2Ad=0$. $B=b+2Ad=b+2ad$. $C+Ad^2-2Bd=0$. $C=2Bd-Ad^2=2bd+4ad^2-ad^2=2bd+3ad^2$. $D+Bd^2-2Cd=0$. $D=2Cd-Bd^2=4bd^2+6ad^3-bd^2-2ad^3=3bd^2+4ad^3$. The coefficients here increase numerically by the addition of 1 in both the coefficients of each term. The common multiple of each term is (d).

$$\frac{a+bx}{(1-dx)^2}=a+(b+2ad)x+(2bd+3ad^2)x^2+(3bd^2+4ad^3)x^3+(4bd^3+5ad^4)x^4+(5bd^4+6ad^5), \text{ \&c.}$$

Example 10.

$$\frac{1+x}{(1-x)^3}=\frac{1+x}{1-3x+3x^2-x^3}=A+Bx+Cx^2+Dx^3+Ex^4, \text{ \&c.}$$

Clearing of fractions, $1+x=A-3Ax+3Ax^2-Ax^3+Bx-3Bx^2+3Bx^3-Bx^4+Cx^2-3Cx^3+3Cx^4-Cx^5+Dx^3$, &c. Trans. $(1+x)$, then $0=(A-1)+(B-1-3A)x+(C-3B+3A)x^2+(D-3C+3B-A)x^3$, &c. $A-1=0$. $A=1$. $B-1-3A=0$. $B=3A+1=3+1=4$. $C-3B+3A=0$. $C=3B-3A=12-3=9$. $D-3C+3B-A=0$. $D=3C-3B+A=27-12+1=16$. The law of increase in the coefficients is as follows: 4th=3(3d)−3(2d)+1st. It will be found that this law gives us the squares of the natural numbers, 1, 2, 3, 4, 5, &c. in their regular order.

The numbers are,	1,	2,	3,	4,	5,	6,	7,	&c.
Their squares are,	1,	4,	9,	16,	25,	36,	49,	&c.

Then $\frac{1+x}{(1-x)^3}=1+4x+9x^2+16x^3+25x^4+36x^5+49x^6$, &c.

SUMMATION OF SERIES.

The examples under this head are so simple, that we omit them. With one or two exceptions, they are all performed in the Algebra, and the student can readily see the manner of solution.

RECURRING SERIES.

Example 1.

The sum of the series, $1+6x+12x^2+48x^3+120x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & 6x+ & 12x^2+ & 48x^3+ & 120x^4, \text{ \&c.}\end{matrix}\right\}$$ By formula, Art. 493,

$$m=\frac{DC-BE}{CC-BD}=\frac{48\times12-6\times120}{12^2-6\times48}=\frac{576-720}{144-288}=\frac{-144}{-144}=1.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{12\times120-48^2}{12^2-6\times48}=\frac{1440-2304}{144-288}=\frac{-864}{-144}=6.$$

$m=1$. $n=6$. $A=1$. $B=6x$. Scale of relation $=1+6$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+6x-x}{1-x-6x^2}=\frac{1+5x}{1-x-6x^2}.$$

Example 2.

The sum of the series, $1+3x+4x^2+7x^3+11x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & 3x+ & 4x^2+ & 7x^3+ & 11x^4, \text{ \&c.}\end{matrix}\right\}$$ By the formula, we have

$$m=\frac{DC-BE}{CC-BD}=\frac{7\times4-3\times11}{4^2-3\times7}=\frac{28-33}{16-21}=\frac{-5}{-5}=1.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{4\times11-7^2}{4^2-3\times7}=\frac{44-49}{16-21}=\frac{-5}{-5}=1.$$

$m=1$. $n=1$. $A=1$. $B=3x$. Scale of relation $=1+1$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+3x-x}{1-x-x^2}=\frac{1+2x}{1-x-x^2}.$$

EXAMPLE 3.

The sum of the series, $1+x+5x^2+13x^3+41x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & x+ & 5x^2+ & 13x^3+ & 41x^4, \text{ \&c.} \end{matrix}\right\}$$ By the formula, we have

$$m=\frac{DC-BE}{CC-BD}=\frac{13\times5-41}{5^2-13}=\frac{65-41}{25-13}=\frac{24}{12}=2.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{5\times41-13^2}{5^2-13}=\frac{205-169}{25-13}=\frac{36}{12}=3.$$

$m=2$. $n=3$. $A=1$. $B=x$. Scale of relation $=2+3$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+x-2x}{1-2x-3x^2}=\frac{1-x}{1-2x-3x^2}.$$

EXAMPLE 4.

The sum of the series, $1+2x+3x^2+4x^3+5x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & 2x+ & 3x^2+ & 4x^3+ & 5x^4, \text{ \&c.} \end{matrix}\right\}$$ By the formula, we have

$$m=\frac{DC-BE}{CC-BD}=\frac{4\times3-2\times5}{3^2-4\times2}=\frac{12-10}{9-8}=\frac{2}{1}=2.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{3\times5-4^2}{3^2-2\times4}=\frac{15-16}{9-8}=\frac{-1}{1}=-1.$$

$m=2$. $n=-1$. $A=1$. $B=2x$. Scale of relation $=2-1$.

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+2x-2x}{1-2x+x^2}=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}.$$

EXAMPLE 5.

The sum of the series, $1+3x+5x^2+7x^3+9x^4$, &c.

$$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & 3x+ & 5x^2+ & 7x^3+ & 9x^4. \end{matrix}\right\}$$ By the formula, we have

$$m=\frac{DC-BE}{CC-BD}=\frac{7\times5-3\times9}{5^2-3\times7}=\frac{35-27}{25-21}=\frac{8}{4}=2.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{5\times9-7^2}{5^2-3\times7}=\frac{45-49}{25-21}=\frac{-4}{4}=-1.$$

$m=2.\quad n=-1.\quad A=1.\quad B=3x.\quad$ Scale of relation $=2-1.$

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+3x-2x}{1-2x+x^2}=\frac{1+x}{1-2x+x^2}=\frac{1+x}{(1-x)^2}.$$

EXAMPLE 6.

The sum of the series, $1+2x+8x^2+28x^3+100x^4$, &c.

$\left\{\begin{matrix} A & B & C & D & E \\ 1+ & 2x+ & 8x^2+ & 28x^3+ & 100x^4, \text{ \&c.}\end{matrix}\right\}$ By the formula, we have

$$m=\frac{DC-BE}{CC-BD}=\frac{28\times 8-2\times 100}{8^2-2\times 28}=\frac{224-200}{64-56}=\frac{24}{8}=3.$$

$$n=\frac{CE-DD}{CC-BD}=\frac{8\times 100-28^2}{8^2-2\times 28}=\frac{800-784}{64-56}=\frac{16}{8}=2.$$

$m=3.\quad n=2.\quad A=1.\quad B=2x.\quad$ Scale of relation $=3+2.$

$$S=\frac{A+B-Amx}{1-mx-nx^2}=\frac{1+2x-3x}{1-3x-2x^2}=\frac{1-x}{1-3x-2x^2}.$$

EXAMPLES WHERE THE SCALE OF RELATION CONSISTS OF *three* PARTS.

EXAMPLE 1.

The sum of the series, $1+4x+6x^2+11x^3+28x^4$, &c.

$A=1.\quad B=4x.\quad C=6x^2.\quad m=2.\quad n=-1.\quad r=3.\quad$ By the formula,

$$S=\frac{A+B+C-(A+B)mx-Anx^2}{1-mx-nx^2-rx^3}=\frac{1+4x+6x^2-2x-8x^2+x^2}{1-2x+x^2-3x^3}$$

$$=\frac{1+2x+x^2-2x^2}{1-2x+x^2-3x^3}=\frac{(1+x)^2-2x^2}{(1-x)^2-3x^3}.$$

EXAMPLE 2.

The sum of the series, $1+x+2x^2+2x^3+3x^4+3x^5$, &c.

$A=1.\quad B=x.\quad C=2x^2.\quad m=1.\quad n=1.\quad r=-1.\quad$ By the formula,

$$S=\frac{A+B+C-(A+B)mx-Anx^2}{1-mx-nx^2-rx^3}=\frac{1+x+2x^2-x-x^2-x^2}{1-x-x^2+x^3}$$
$$=\frac{1}{1-x-x^2+x^3}.$$

METHOD OF DIFFERENCES.

Example 1.

The nth term of the series, 1, 3, 6, 10, 15, 21, &c.

Proposed series,	1, 3, 6, 10, 15, 21, &c.
First order of diff.	2, 3, 4, 5, 6, &c.
Second do.	1, 1, 1, 1, &c.
Third do.	0, 0, 0.

$a=1$. $D'=2$. $D''=1$. $D'''=0$. By formula, Art. 493, nth term $=1+(n-1)2+(n-1)\frac{n-2}{2}\times 1=1+2n-2+\frac{n^2-3n+2}{2}=\frac{1}{2}(2+4n-4+n^2-3n+2)=\frac{1}{2}(n^2+n)$. $20th$ term $=\frac{1}{2}(20^2+20)=210$. $50th$ term $=\frac{1}{2}(50^2+50)=1275$.

Example 2.

The $20th$ term of the series, 1^3, 2^3, 3^3, 4^3, 5^3, &c.

Proposed series,	1, 8, 27, 64, 125, &c.
First order of diff.	7, 19, 37, 61, &c.
Second do.	12, 18, 24, &c.
Third do.	6, 6, &c.
Fourth do.	0.

$a=1$. $D'=7$. $D''=12$. $D'''=6$. $D''''=0$. $n=20$. $20th$ term $=1+(20-1)7+(20-1)\frac{20-2}{2}\times 12+(20-1)\frac{20-2}{2}\times\frac{20-3}{3}\times 6=1+19\times 7+19\times 9\times 12+19\times 9\times\frac{17}{3}\times 6=1+133+2052+5814=8000$.

Example 3.

The 12*th* term of the series, 2, 6, 12, 20, 30, &c.

Proposed series,	2, 6, 12, 20, 30, &c.
First order of diff.	4, 6, 8, 10, &c.
Second do.	2, 2, 2, &c.
Third do.	0, 0.

$a=2$. $D'=4$. $D''=2$. $D'''=0$. $n=12$. 12th term $=$ $2+(12-1)4+(12-1)\frac{12-2}{2}\times 2=2+11\times 4+11\times 5\times 2=$ $2+44+110=156$.

Example 4.

The 15*th* term of the series, 1^2, 2^2, 3^2, 4^2, 5^2, &c.

Proposed series,	1, 4, 9, 16, 25, &c.
First order of diff.	3, 5, 7, 9, &c.
Second do.	2, 2, 2, &c.
Third do.	0, 0.

$a=1$. $D'=3$. $D''=2$. $D'''=0$. $n=15$. The 15th term $=1+(15-1)3+(15-1)\frac{15-2}{2}\times 2=1+14\times 3+14\times \frac{13}{2}\times 2$ $=1+42+182=225$.

EXAMPLES OF THE *sum* OF *n* TERMS OF THE SERIES.

Example 1.

The sum of n terms of the series, 1, 3, 5, 7, 9, &c.

Proposed series,	1, 3, 5, 7, 9, &c.
First order of diff.	2, 2, 2, 2, &c.
Second do.	0, 0, 0.

$a=1$. $D'=2$. $D''=0$. By the second formula, $S=n+n\frac{n-1}{2}\times 2=n+n(n-1)=n+n^2-n=n^2$.

EXAMPLE 2.

The sum of n terms of the series, 1^2, 2^2, 3^2, 4^2, 5^2, &c.

Proposed series,	1, 4, 9, 16, 25, &c.
First order of diff.	3, 5, 7, 9, &c,
Second do.	2, 2, 2, &c.
Third do.	0, 0.

$a=1$. $D'=3$. $D''=2$. $D'''=0$. $S=n+n\left(\frac{n-1}{2}\right)\times 3+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 2$. Reducing, $S=n+\frac{3}{2}(n^2-n)+\frac{1}{3}(n^3-3n^2+2n)$. Clearing of fractions, $6S=6n+9n^2-9n+2n^3-6n^2+4n=2n^3+3n^2+n$. Div. by 6, then $S=\frac{1}{6}(2n^3+3n^2+n)$. If $n=20$, then $S=\frac{1}{6}(2\times 20^3+3\times 20^2+20)=\frac{1}{6}(16000+1200+20)=2870$.

EXAMPLE 3.

The sum of n terms of the series, 1^3, 2^3, 3^3, 4^3, 5^3, &c.

Proposed series,	1, 8, 27, 64, 125, &c.
First order of diff.	7, 19, 37, 61, &c.
Second do.	12, 18, 24, &c.
Third do.	6, 6, &c.
Fourth do.	0.

$a=1$. $D'=7$. $D''=12$. $D'''=6$. $D''''=0$. Then $S=n+n\left(\frac{n-1}{2}\right)\times 7+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 12+n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times\left(\frac{n-3}{4}\right)\times 6$. Reducing, we have $S=n+\frac{7}{2}(n^2-n)+(n^3-3n^2+2n)\times 2+(n^4-6n^3+11n^2-6n)\times\frac{1}{4}$. $4S=4n+14n^2-14n+8n^3-24n^2+16n+n^4-6n^3+11n^2-6n$. Reducing, $4S=n^4+2n^3+n^2$; and $S=\frac{1}{4}(n^4+2n^3+n^2)$. If n is equal to 50, then $S=\frac{1}{4}(50^4+2\times 50^3+50^2)$. Expanding, $S=\frac{1}{4}(6250000+250000+2500)=1625625$.

EXAMPLE 4.

The sum of n terms of the series, 2, 6, 12, 20, 30, &c.

Proposed series,	2, 6, 12, 20, 30, &c.
First order of diff.	4, 6, 8, 10, &c.
Second do.	2, 2, 2, &c.
Third do.	0, 0.

$a=2$. $D'=4$. $D''=2$. $D'''=0$. $S=2n+n\left(\frac{n-1}{2}\right)\times 4 +n\left(\frac{n-1}{2}\right)\times\left(\frac{n-2}{3}\right)\times 2$. Reducing, $S=2n+(n^2-n)\times 2+\frac{1}{3}(n^3-3n^2+2n)$. Clearing of fractions, $3S=6n+6n^2-6n+n^3-3n^2+2n=n^3+3n^2+2n=n(n^2+3n+2)$. But $(n^2+3n+2)=(n+1)\times(n+2)$. Then $3S=n(n+1)\times(n+2)$. Dividing by 3, then $S=\frac{1}{3}n(n+1)\times(n+2)$.

EXAMPLE 5.

The sum of 20 terms of the series, 1, 3, 6, 10, 15, &c.

Proposed series,	1, 3, 6, 10, 15, &c.
First order of diff.	2, 3, 4, 5, &c.
Second do.	1, 1, 1, &c.
Third do.	0, 0.

$a=1$. $D'=2$. $D''=1$. $D'''=0$. $n=20$. $S=20+20\left(\frac{20-1}{2}\right)\times 2+20\left(\frac{20-1}{2}\right)\times\left(\frac{20-2}{3}\right)\times 1$. Reducing, $S=20+20\times 19+20\times 19\times 18\times\frac{1}{6}=20+380+1140=1540$.

EXAMPLE 6.

The sum of 12 terms of the series, 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, &c.

Proposed series,	1, 16, 81, 256, 625, 1296, &c.
First order of diff.	15, 65, 175, 369, 671, &c.
Second do.	50, 110, 194, 302, &c.
Third do.	60, 84, 108, &c.
Fourth do.	24, 24, &c.
Fifth do.	0.

$a=1$. $D'=15$. $D''=50$. $D'''=60$. $D''''=24$. $D'''''=0$. $n=12$. $S=12+12\left(\frac{12-1}{2}\right)\times 15+12\left(\frac{12-1}{2}\right)\times\left(\frac{12-2}{3}\right)\times 50+12\left(\frac{12-1}{2}\right)\times\left(\frac{12-2}{3}\right)\times\left(\frac{12-3}{4}\right)\times 60+12\left(\frac{12-1}{2}\right)\times\left(\frac{12-2}{3}\right)\times\left(\frac{12-3}{4}\right)\times\left(\frac{12-4}{5}\right)\times 24$.
$S=12+12\times\frac{11}{2}\times 15+12\times\frac{11}{2}\times\frac{10}{3}\times 50+12\times\frac{11}{2}\times\frac{10}{3}\times\frac{9}{4}\times 60+12\times\frac{11}{2}\times\frac{10}{3}\times\frac{9}{4}\times\frac{8}{5}\times 24$. Reducing this expression, $S=12+990+11000+29700+190008=231710$.

RESOLUTION OF HIGHER EQUATIONS.

Example 2.

Required the roots of the equation $x^3-8x^2+4x+48=0$ Suppose one of them to be 4.1 or 4.2. Then

$\begin{cases}(4.1)^3-8(4.1)^2+4(4.1)+48=1.059; \text{ or expanding,}\\ 68.921-134.48+16.4+48=1.059, \text{ or the first error.}\end{cases}$

$\begin{cases}(4.2)^3-8(4.2)^2+4(4.2)+48=2.232; \text{ or expanding,}\\ 74.088-141.12+16.8+48=2.232, \text{ or the second error.}\end{cases}$

Difference of errors $=2.232-1.059=1.173$. Then $1.173 : 0.1 :: 1.159 : 0.1$, very nearly. $4.1-0.1=4$.

We have $4^3-8\times 4^2+4\times 4+48=0$, and verifies the conditions. As $x-4=0$, dividing the given equation by it, $(x^3-8x^2+4x+48)\div(x-4)=x^2-4x-12=0$. Trans. $x^2-4x=12$. Completing the square, $x^2-4x+4=16$. Extracting the root, $x-2=\pm 4$. $x=+6$ or ± 2. The conditions make the 2 minus; for $4+6-2=8$, and if the 2 were plus, this would not be true. Roots $=-2+4+6$.

Example 3.

Required the roots of the equation $x^3-16x^2+65x-50=0$. Suppose one of them to be 5.1 or 5.2. Then

$\begin{cases} (5.1)^3-16(5.1)^2+65(5.1)-50=2.009; \text{ or expanding,} \\ 132.651-416.16+331.5-50=2.009, \text{ the first error.} \end{cases}$

$\begin{cases} (5.2)^3-16(5.2)^2+65(5.2)-50=4.032; \text{ or expanding,} \\ 140.608-432.64+338-50=4.032, \text{ the second error.} \end{cases}$

Difference of errors $=4.32-2.009=2.023$. Then $2.023:0.1::2.009:0.099$, very nearly. $5.1-0.099=5.001$. Let 5.001 or 5.002 be the assumed number.

$\begin{cases} (5.001)^3-16(5.001)^2+65(5.001)-50=0.02; \text{ or expanding,} \\ 125.075-400.16+325.065-50=0.02, \text{ the first error.} \end{cases}$

$\begin{cases} (5.002)^3-16(5.002)^2+65(5.002)-50=0.04; \text{ or expand.} \\ 125.15-400.32+325.13-50=0.04, \text{ the second error.} \end{cases}$

Difference of errors $=0.04-0.02=0.02$. Then $0.02:0.001::0.02:0.001$. And $5.001-0.001=5$. Then $5^3-16\times5^2+65\times5-50=0$, and verifies the conditions. As $x-5=0$, dividing the given equation by it, $(x^3-16x^2+65x^2-50)\div(x-5)=x^2-11x+10=0$. Trans. $x^2-11x=-10$. Completing the square by the second method, $4x^2-x+(11)^2=(11)^2-40=81$. Extracting the root, $2x-11=\sqrt{81}=9$. $2x=\pm9+11=20$, or 2. $x=10$ or 1. Roots $=1+5+10$. And $1+5+10=16$. $1\times5\times10=50$.

Example 4.

Required the roots of the equation $x^3+2x^2-33x=90$. Trans. 90, then $x^3+2x^2-33x-90=0$. Suppose one of them to be 6.1 or 6.2; then

$\begin{cases} (6.1)^3+2(6.1)^2-33(6.1)-90=10.101; \text{ or expanding,} \\ 226.981+74.42-201.3-90=10.101, \text{ or the first error.} \end{cases}$

$\begin{cases} (6.2)^3+2(6.2)^2-33(6.2)-90=20.608; \text{ or expanding,} \\ 238.328+76.88-204.6-90=20.608, \text{ the second error.} \end{cases}$

Difference of errors $=20.608-10.101=10.507$. Then $10.507:0.1::10.101:0.096$. $6.1-0.096=6.004$. Let 6.004 or 6.005 be the assumed number.

$\begin{cases} (6.004)^3+2(6.004)^2-33(6.004)-90=0.396; \text{ or expanding,} \\ 216.432+72.096-198.132-90=0.396, \text{ the first error.} \end{cases}$

$\begin{cases} (6.005)^3+2(6.005)^2-33(6.005)-90=0.495; \text{ or expanding,} \\ 216.54+72.12-198.165-90=0.495, \text{ the second error.} \end{cases}$

Difference of errors $=0.495-0.396=0.099$. Then $0.099 : 0.001 :: 0.396 : 0.004$. And $6.004-0.004=6$. $6^3+2\times6^2-33\times6-90=0$, and verifies the conditions. As $x-6=0$, dividing the given equation by it, $(x^3+2x^2-33x-90)\div(x-6)=x^2+8x+15=0$. Transposing, $x^2+8x=-15$. Completing the square, $x^2+8x+16=16-15=1$. Evolving, $x+4=1$. Transposing, $x=\pm1-4=-3$ or -5. Roots $=6-3-5$. $6-3-5=-2$. $6\times-3\times-5=90$.

Example 5.

Required a near value of one of the roots of the equation $x^3+9x^2+4x=80$. Transposing, $x^3+9x^2+4x-80=0$. Suppose one of them to be 2.5 or 2.6. Then

$\begin{cases} (2.5)^3+9(2.5)^2+4(2.5)-80=1.875; \text{ or expanding,} \\ 15.625+56.25+10-80=1.875, \text{ the first error.} \end{cases}$

$\begin{cases} (2.6)^3+9(2.6)^2+4(2.6)-80=8.816; \text{ or expanding,} \\ 17.576+60.84+10.4-80=8.816, \text{ the second error.} \end{cases}$

Difference of errors $=8.816-1.875=6.941$. Then $6.941 : 0.1 :: 1.875 : 0.027$. And $2.5-0.027=2.473$. $(2.473)^3+9(2.473)^2+4(2.473)-80=0.05776$. Thus 2.473 is very nearly one of the roots.

Example 6.

Required a near value of one of the roots of the equation $x^3+x^2+x=100$. Trans. $x^3+x^2+x-100=0$. Suppose one of the numbers to be 4.3 or 4.4. Then

$\begin{cases} (4.3)^3+(4.3)^2+(4.3)-100=2.297; \text{ or expanding,} \\ 79.507+18.49+4.3-100=2.297, \text{ the first error.} \end{cases}$

$\begin{cases} (4.4)^3+(4.4)^2+4.4-100=8.944; \text{ or expanding,} \\ 85.184+19.36+4.4-100=8.944, \text{ the second error.} \end{cases}$

Difference of errors $=8.944-2.297=6.647$. Then $6.647 : 0.1 :: 2.297 : 0.034+$. $4.3-0.034+=4.266-$.

Again, suppose 4.266 or 4.265 to be one of the numbers.

$\begin{cases} (4.266)^3+(4.266)^2+(4.266)-100=0.101; \text{ or expanding,} \\ 77.636+18.199+4.266-100=0.101, \text{ the first error.} \end{cases}$

$\begin{cases} (4.265)^3+(4.265)^2+(4.265)-100=0.036; \text{ or expanding,} \\ 77.581+18.19+4.265-100=0.036, \text{ the second error.} \end{cases}$

Diff. of errors $=0.101-0.036=0.065$. Then $0.065:0.001::0.036:0.00055+$. And $4.265-0.00055+=4.26445-$ $=$ nearly the required root.

$\begin{cases} (4.26445)^3+(4.26445)^2+(4.26445)-100=0.00126. \\ \text{Expanded, } 77.55128+18.18553+4.26445-100=0.00126. \end{cases}$

As this lacks but a thousandth of the given number, it is sufficiently near, as the *exact* root cannot be obtained.

Here we will take leave of our readers, hoping that they may have derived some little assistance in their studies from this treatise. And, thanking them for all the patience which they may have had to exercise in laboring through our explanations, as well as begging their pardon for the blunders which the printer or our own hands have unwittingly committed, we would say *Au Revoir*.

THE END.